Finding minimal obstructions to graph colouring

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Definition (*k*-colouring)

A **k-(vertex)-colouring** of a graph G is an assignment of colours $\{1, 2, ..., k\}$ to the vertices of G such that any two adjacent vertices receive a distinct colour.

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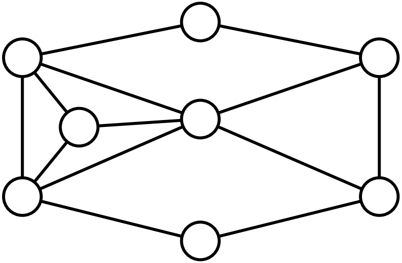

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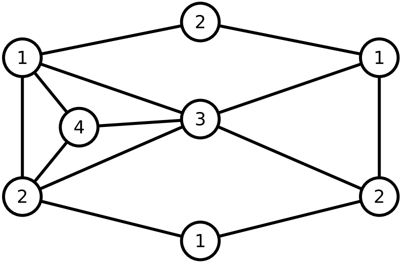

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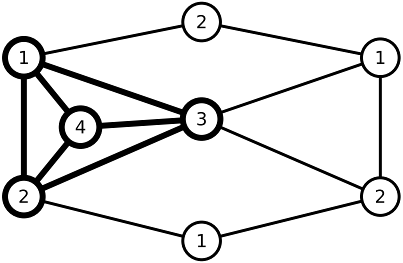

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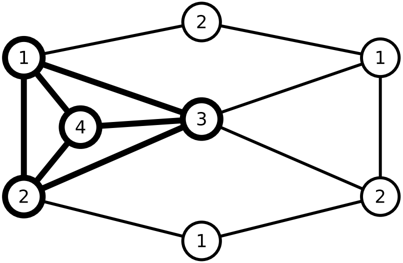

- We call K_4 an **obstruction** for 3-colourability.

Complexity of graph colouring

- The problem of deciding if a graph is k-colourable is called the **k-colourability problem**.
- This is a very hard problem:
  - It is NP-complete.
  - Even if k is fixed (for every $k \geq 3$).

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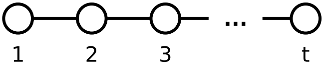
- This motivates the study of k-colouring of P_t-free graphs.
- P_t is the path on t vertices.



k-colourability in P_t-free graphs

Definition (k-critical H-free graph)

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- Determine finite sets S of all $(k+1)$-critical P_t-free graphs (at least if such a finite set exists).
- This means: a P_t-free graph is k-colourable $\iff$ it does not contain a graph from S as subgraph.

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- If S is finite this gives a polynomial algorithm to test if a P_t-free graph is k-colourable.
- Moreover it provides a **no-certificate** for k-colourability.
  - Cf. Kuratowski graphs as no-certificate for planarity.

k-colourability in P_t-free graphs

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There are six 4-critical P_5-free graphs.

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- **Golovach et al., 2014:** is there a certifying algorithm for 3-colourablity of P_6-free graphs?
- **Seymour, 2014:** for which connected graphs H is the set of 4-critical H-free graphs finite?

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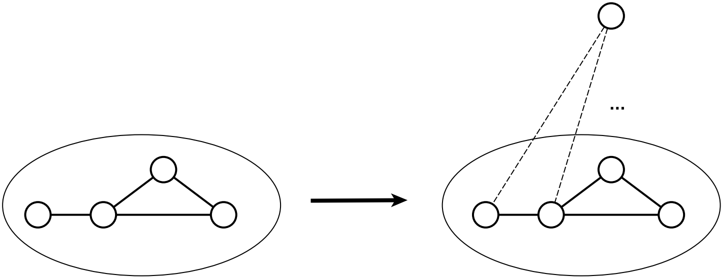
Our approach:

- Computer approach through graph generation:
  - Design generation algorithm for k-critical P_t-free graphs.

Basic construction operation

The following operation can be used to construct any simple graph:



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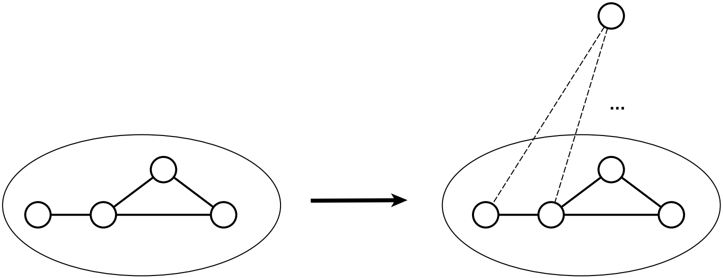


- Create a new vertex and connect it to the vertices of the parent graph in all possible ways.
- Note: all graphs constructed from *G* contain *G* as induced subgraph!

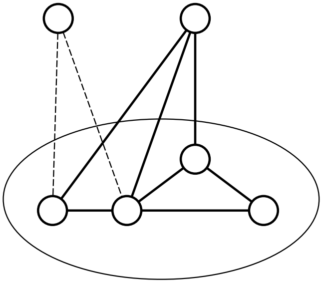
Properties of k-critical graphs

Theorem (Folklore)

Critical graphs do not contain similar vertices.

Definition (Similar vertices)

Similar vertices are vertices $v, w \in V(G) : N(v) \subseteq N(w)$.



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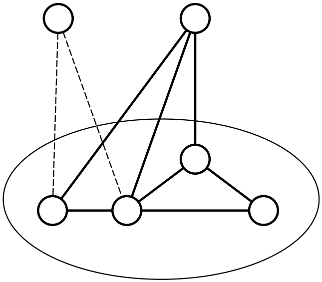

- Note: the above theorem can also be generalised to *similar* edges, triangles, P_3's, C_4's etc.

Properties of k-critical graphs

Theorem (Hidden edge)

If a graph G contains two non-adjacent vertices u, v such that for all neighbours x of u it holds that v cannot receive the same colour as x in any $(k-1)$-colouring of $G-u$, then G cannot be k-critical.

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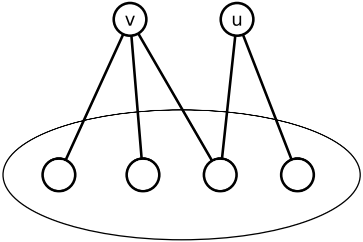

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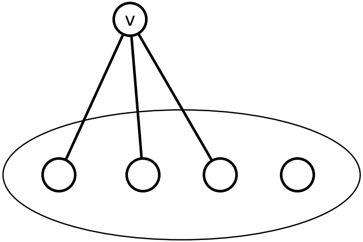

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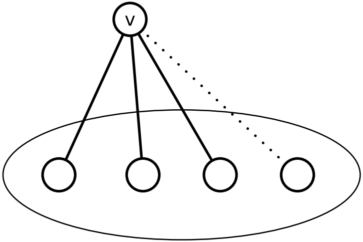

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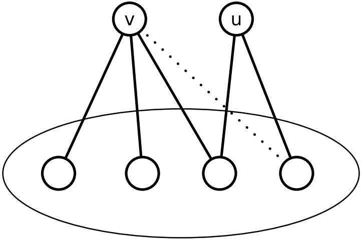

Specialised construction algorithm

if Graph is P_t-free AND not generated before [nauty] **then**
  if Graph is not $(k-1)$-colourable **then**
    if Graph is k-critical **then**
      Output graph
    end if
  else
    // Apply expansions: see next slide...
  end if
end if

Specialised construction algorithm

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if Graph is P_t-free AND not generated before [nauty] then
    if Graph is not (k − 1)-colourable then
        if Graph is k-critical then
            Output graph
        end if
    else
        // Apply expansions:
        if Graph contains similar vertices then
            Destroy the *best* similar vertex in all possible ways
        else if Graph contains similar vertices including hidden edges then
            Destroy the *best* similar vertex in all possible ways
        else if Graph contains similar edges then
            Destroy the *best* similar edge in all possible ways
        else if Graph contains ... (i.e. try to apply other lemmas) then
            Destroy ... in all possible ways
        else
            Apply all possible expansions
        end if
    end if
end if
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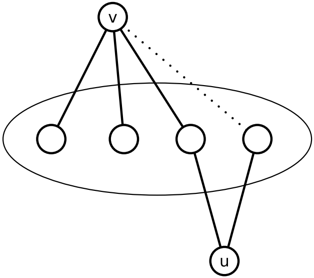
Destroying similar vertices

How to destroy similar vertices u, v of G (with $N(u) \subseteq N(v)$)?



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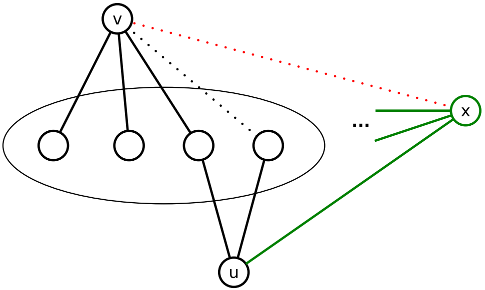
How to destroy similar vertices u, v of G (with $N(u) \subseteq N(v)$)?



- Connect new vertex x to u but not to v (in all possible ways).
- Make sure there is no hidden edge between x and v in $G - u + x$ (helps a lot in most cases!).

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- We generated all 4-critical P_6-free graphs up to 28 vertices (this took approximately 9 CPU years)...
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Theorem

There are 24 4-critical P_6-free graphs with at most 28 vertices.

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 1 graphs : n=4
 1 graphs : n=6
 2 graphs : n=7
 3 graphs : n=8
 4 graphs : n=9
 6 graphs : n=10
 2 graphs : n=11
 1 graphs : n=12
 3 graphs : n=13
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24 graphs altogether;
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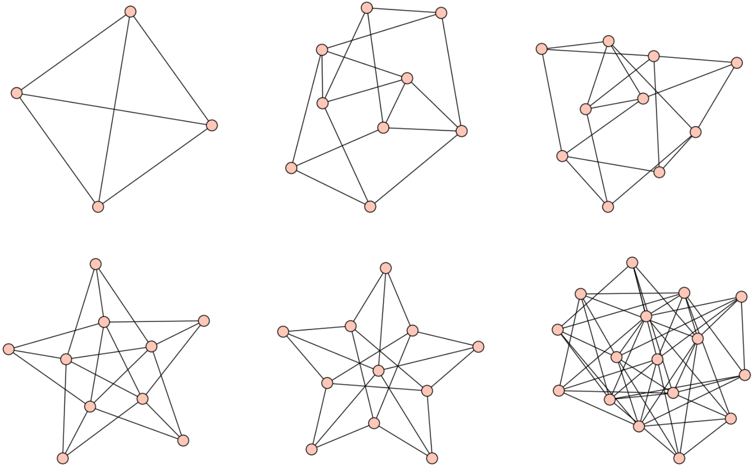
So it is VERY likely that these are the only 4-critical P_6-free graphs...

Results – 4-critical (P_6, diamond)-free

Theorem

There are 6 4-critical (P_6, diamond)-free graphs.



Results – 4-critical P_6-free

By combining:

- Knowledge of all 4-critical (P_6, diamond)-free graphs (computer-aided).
- Knowledge of all 4-critical P_6-free graphs up to 28 vertices (computer-aided).
- Extensive structural analysis by hand (see arXiv paper for details).

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There are 24 4-critical P_6-free graphs.

Moreover, given a connected graph H, there are finitely many 4-critical H-free graphs if and only if H is a subgraph of P_6.

Results – 4-critical P_6-free: graphs 1 to 12

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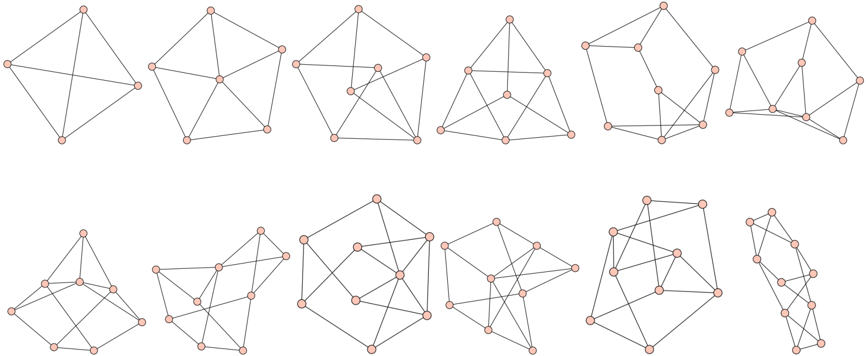

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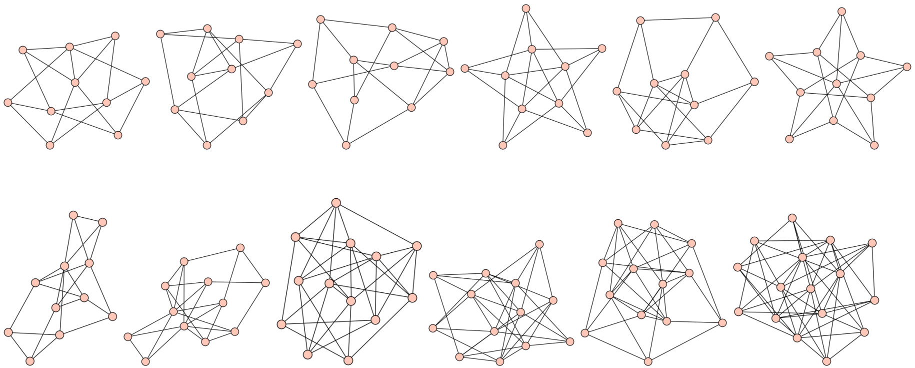

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Other ongoing work:

- Obstructions for list k-colourability.
- Each vertex v has a list $L(v) \subseteq \{1, ..., k\}$ with available colours.

Thanks for your attention!