On Longest Cycles in Essentially 4-connected Planar Graphs

J. Harant

TU Ilmenau, Germany

(joint work with I. Fabrici and S. Jendrol’, Košice, Slovakia)

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All graphs $G$ considered here are *polyhedral*, i.e. planar and 3-connected.

- $n = n(G)$ - order of $G$
- $\text{circ}(G)$ - length of a longest cycle of $G$ ($\text{circumference}$ of $G$)
- If $\text{circ}(G) = n$, then $G$ is *hamiltonian* and a longest cycle is a *hamiltonian cycle*.
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There are infinitely many maximal planar graphs $G$ with

$$\text{circ}(G) \leq 9n(G)^{\log_3 2} \quad (\log_3 2 = 0.6309...).$$

- Is the exponent $\log_3 2$ smallest possible for maximal planar graphs?
- Can $\log_3 2$ be decreased if arbitrary polyhedral graphs are considered?
- Later the coefficient 9 was decreased several times.
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A graph $G$ is essentially 4-connected if $G$ is 3-connected and each 3-separator forms the neighborhood of a vertex of degree 3.
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A graph \( G \) is essentially 4-connected if \( G \) is 3-connected and each 3-separator forms the neighborhood of a vertex of degree 3.
Let $G$ be polyhedral and essentially 4-connected.

- $\text{circ}(G) \geq \frac{2n(G)+4}{5}$.
  (Jackson, Wormald 1992)

- $\text{circ}(G) \geq \frac{3}{4}n(G)$ if $G$ is cubic.
  (Grüenbaum, Malkevitch 1976, Zhang 1987)

- If $c > \frac{2}{3}$, then there is an infinite family of graphs $G$ such that $\text{circ}(G) \leq c \cdot n(G)$.
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- The last statement is even true for maximal planar graphs.
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- The last statement is even true for maximal planar graphs.
A 4-connected maximal planar graph $G'$ on 32 vertices.
2 \times 32 - 4 = 60 \text{ red vertices}
A essentially 4-connected maximal planar graph $G$ on $32 + 2 \times 32 - 4 = 92$ vertices.
- $G$ has 32 black vertices
- the red vertices are independent
- a longest cycle of $G$ has at most $2 \times 32 = 64$ vertices
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Infinitely many essentially 4-connected maximal planar graphs $G$ with $\text{circ}(G) \leq \frac{2n(G)+8}{3}$

- $G'$ - a 4-connected maximal plane graph on $n'$ vertices.
- $G$ - obtained from $G'$ by inserting a new vertex into each face of $G'$ and connecting it with the tree boundary vertices of that face by an edge.
- $G$ is an essentially 4-connected maximal plane graph on $n = n' + (2n' - 4)$ vertices.
- The $2n' - 4$ vertices in $V(G) \setminus V(G')$ are pairwise independent.
- Hence, each cycle of $G$ contains at most $2n' = \frac{2n+8}{3}$ vertices.
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- $\text{circ}(G) \geq \frac{3}{4}n(G)$ if $G$ is cubic - Grünbaum, Malkevitch 1976, Zhang 1987
  There is an infinite family of essentially 4-connected cubic planar graphs $G$ such that $\text{circ}(G) \leq \frac{76}{77}n(G)$.


(i) $\text{circ}(G) \geq \frac{n(G)+4}{2}$.

(ii) $\text{circ}(G) \geq \frac{3}{5}n(G)$ if $\Delta = 4$.

(iii) $\text{circ}(G) \geq \frac{13}{21}(n(G) + 4)$ if $G$ is maximal planar.
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Lemma: If $G$ is an essentially 4-connected planar graph, then $G$ contains an OI3-cycle $C$.

Let $C$ be a longest OI3-cycle of $G$.

For each edge $xy$ of $C$, $x$ and $y$ do not have a common neighbor in $int(C) \cap V(G)$.

Lemma: If $C$ is a cycle of a plane graph $G$ on at least 4 vertices such that $int(C) \cap V(G)$ is an independent set of vertices of degree 3 in $G$ and, for each edge $xy$ of $C$, $x$ and $y$ do not have a common neighbor in $int(C) \cap V(G)$, then $|int(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.

$n(G) = |V(C)| + |int(C) \cap V(G)| + |ext(C) \cap V(G)| \leq 2|V(C)| - 4$. 
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Sketch of the proof of $\text{circ}(G) \geq \frac{n(G)+4}{2}$.

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- $n(G) = |V(C)| + |\text{int}(C) \cap V(G)| + |\text{ext}(C) \cap V(G)| \leq 2|V(C)| - 4$. 
Lemma: If $C$ is a cycle of a plane graph $G$ on at least 4 vertices such that $\text{int}(C) \cap V(G)$ is an independent set of vertices of degree 3 in $G$ and, for each edge $xy$ of $C$, $x$ and $y$ do not have a common neighbor in $\text{int}(C) \cap V(G)$, then $|\text{int}(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.

- If $|\text{int}(C) \cap V(G)| = 0$, then nothing is to prove.
- Induction on $|V(C)|$.
- If $|V(C)| \leq 5$, then, obviously, $|\text{int}(C) \cap V(G)| = 0$. 

![Diagram of cycle C](image-url)
Proof of Lemma: If $C$ is a cycle of a plane graph $G$ on at least 4 vertices such that $\text{int}(C) \cap V(G)$ is an independent set of vertices of degree 3 in $G$ and, for each edge $xy$ of $C$, $x$ and $y$ do not have a common neighbor in $\text{int}(C) \cap V(G)$, then $|\text{int}(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.

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![Diagram of a cycle](image)
Lemma: If $C$ is a cycle of a plane graph $G$ on at least 4 vertices such that $\text{int}(C) \cap V(G)$ is an independent set of vertices of degree 3 in $G$ and, for each edge $xy$ of $C$, $x$ and $y$ do not have a common neighbor in $\text{int}(C) \cap V(G)$, then $|\text{int}(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.

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Lemma: If $C$ is a cycle of a plane graph $G$ on at least 4 vertices such that $\text{int}(C) \cap V(G)$ is an independent set of vertices of degree 3 in $G$ and, for each edge $xy$ of $C$, $x$ and $y$ do not have a common neighbor in $\text{int}(C) \cap V(G)$, then $|\text{int}(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.

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![Triangle Cycle](image)
Lemma: If \( C \) is a cycle of a plane graph \( G \) on at least 4 vertices such that \( \text{int}(C) \cap V(G) \) is an independent set of vertices of degree 3 in \( G \) and, for each edge \( xy \) of \( C \), \( x \) and \( y \) do not have a common neighbor in \( \text{int}(C) \cap V(G) \), then

\[
|\text{int}(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4).
\]

- \( |V(C)| \geq 6 \) and \( |\text{int}(C) \cap V(G)| > 0 \).

\[
\begin{align*}
|\text{int}(C_i) \cap V(G)| &\leq \frac{|V(C_i)|}{2} - 2 \text{ for } i = 1, 2, 3 \text{ (ind. hyp.)}. \\
|V(C_1)| + |V(C_2)| + |V(C_3)| &= |V(C)| + 6. \\
|\text{int}(C_1) \cap V(G)| + |\text{int}(C_2) \cap V(G)| + |\text{int}(C_3) \cap V(G)| &= |\text{int}(C) \cap V(G)| - 1.
\end{align*}
\]
Lemma: If \( C \) is a cycle of a plane graph \( G \) on at least 4 vertices such that \( \text{int}(C) \cap V(G) \) is an independent set of vertices of degree 3 in \( G \) and, for each edge \( xy \) of \( C \), \( x \) and \( y \) do not have a common neighbor in \( \text{int}(C) \cap V(G) \), then\[ |\text{int}(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4). \]

- \(|V(C)| \geq 6\) and \(|\text{int}(C) \cap V(G)| > 0\).

\[
\begin{align*}
|\text{int}(C_1) \cap V(G)| &\leq \frac{|V(C_i)|}{2} - 2 \text{ for } i = 1, 2, 3 \text{ (ind. hyp.)}. \\
|V(C_1)| + |V(C_2)| + |V(C_3)| &= |V(C)| + 6. \\
|\text{int}(C_1) \cap V(G)| + |\text{int}(C_2) \cap V(G)| + |\text{int}(C_3) \cap V(G)| &= \text{int}(C) \cap V(G) | - 1.
\end{align*}
\]
Proof of

Lemma: If $C$ is a cycle of a plane graph $G$ on at least 4 vertices such that $\text{int}(C) \cap V(G)$ is an independent set of vertices of degree 3 in $G$ and, for each edge $xy$ of $C$, $x$ and $y$ do not have a common neighbor in $\text{int}(C) \cap V(G)$, then $|\text{int}(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.

- $|V(C)| \geq 6$ and $|\text{int}(C) \cap V(G)| > 0$.

- $|\text{int}(C_i) \cap V(G)| \leq \frac{|V(C_i)|}{2} - 2$ for $i = 1, 2, 3$ (ind. hyp.).

- $|V(C_1)| + |V(C_2)| + |V(C_3)| = |V(C)| + 6$.

- $|\text{int}(C_1) \cap V(G)| + |\text{int}(C_2) \cap V(G)| + |\text{int}(C_3) \cap V(G)| = |\text{int}(C) \cap V(G)| - 1$. 
Proof of

Lemma: If $C$ is a cycle of a plane graph $G$ on at least 4 vertices such that $\text{int}(C) \cap V(G)$ is an independent set of vertices of degree 3 in $G$ and, for each edge $xy$ of $C$, $x$ and $y$ do not have a common neighbor in $\text{int}(C) \cap V(G)$, then
\[ |\text{int}(C) \cap V(G)| \leq \frac{1}{2} (|V(C)| - 4). \]

- $|V(C)| \geq 6$ and $|\text{int}(C) \cap V(G)| > 0$.

- $|\text{int}(C_i) \cap V(G)| \leq \frac{|V(C_i)|}{2} - 2$ for $i = 1, 2, 3$ (ind. hyp.).
- $|V(C_1)| + |V(C_2)| + |V(C_3)| = |V(C)| + 6$.
- $|\text{int}(C_1) \cap V(G)| + |\text{int}(C_2) \cap V(G)| + |\text{int}(C_3) \cap V(G)| = |\text{int}(C) \cap V(G)| - 1.$
Thank you for your attention!