Barnette was right: (Not only) fullerene graphs are hamiltonian

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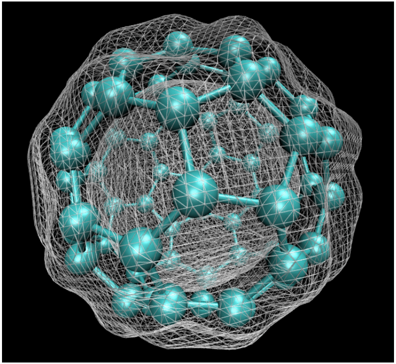
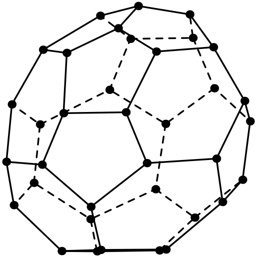
LaBRI / Université de Bordeaux

Gent, 2 August 2016

- *Fullerenes* are carbon molecules, where atoms are arranged in pentagons and hexagons. They can be represented by convex polytopes or by plane graphs.



- A *fullerene graph* is a 3-connected cubic plane graph with pentagonal and hexagonal faces.



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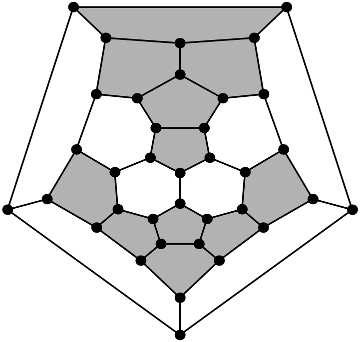

Barnette's conjecture

Fullerene graphs are conjectured to be hamiltonian.
The question is a special case of

Conjecture (Barnette, 1969)

All 3-connected cubic planar graphs with faces of size at most 6 are hamiltonian.

Recall that $3f_3 + 2f_4 + f_5 = 12$ for all graphs from Barnette's conjecture.

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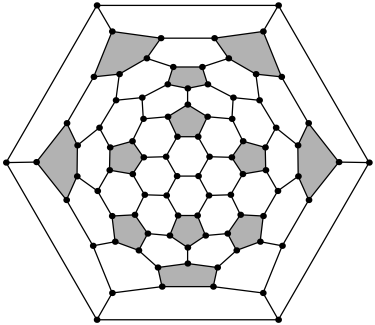
Long cycles in fullerene graphs

Let the length of the longest cycle in a fullerene graph on n vertices be denoted by $\Phi(n)$.

Long cycles in fullerene graphs

Jendrol', Owens (1995): $\Phi(n) \geq \frac{4}{5}n$.

Král', Pangrác, Sereni, Škrekovski (2007): $\Phi(n) \geq \frac{5}{6}n - \frac{2}{3}$.



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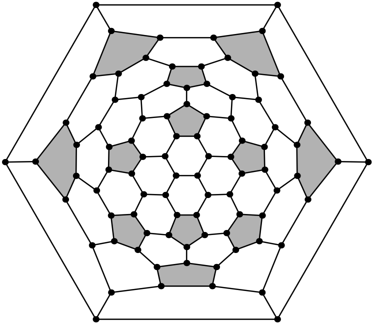

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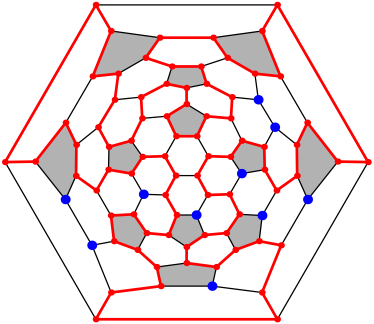

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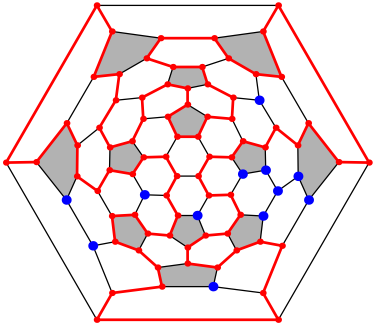

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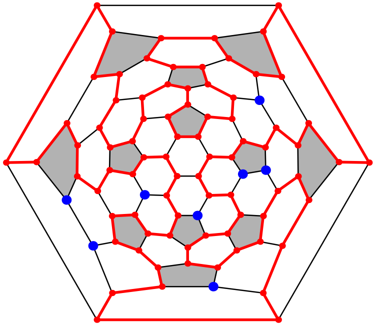

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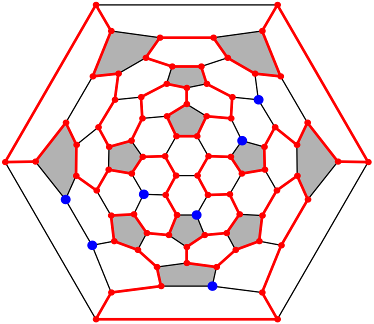

Long cycles in fullerene graphs

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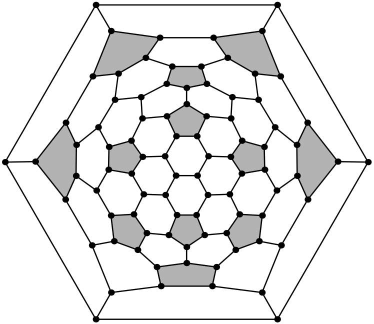

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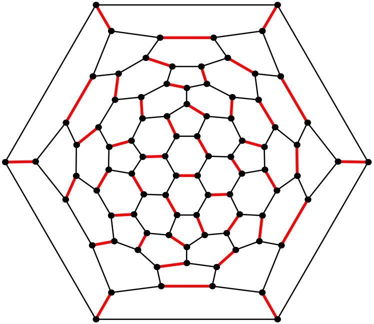

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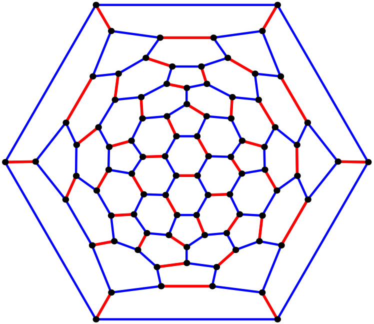

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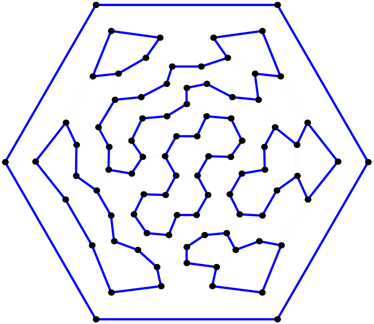

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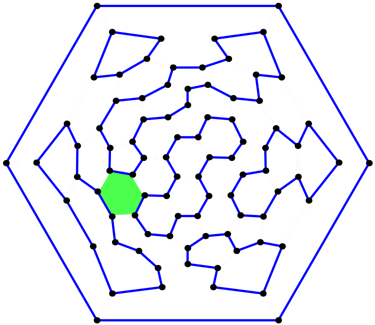

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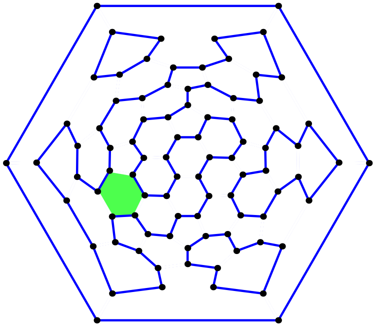

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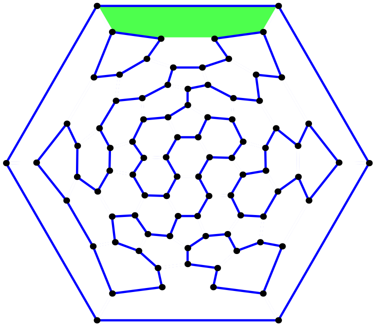

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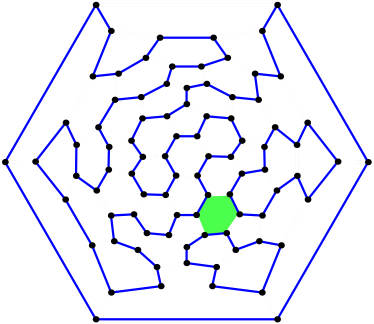

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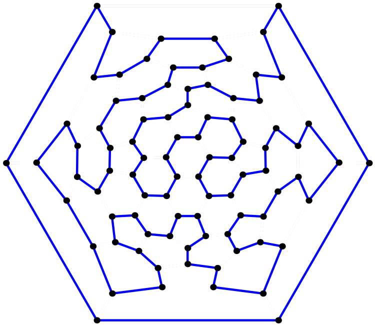
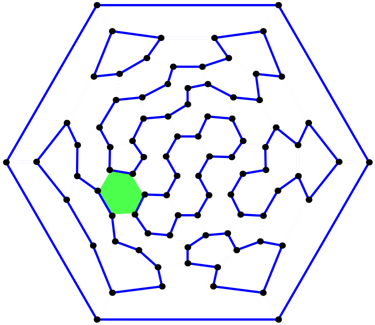


Resonant hexagons in fullerene graphs

A hexagonal face f is called *resonant* with respect to a perfect matching M (or the corresponding 2-factor $G \setminus M$), if precisely three edges incident with f are in M (and three in $G \setminus M$).



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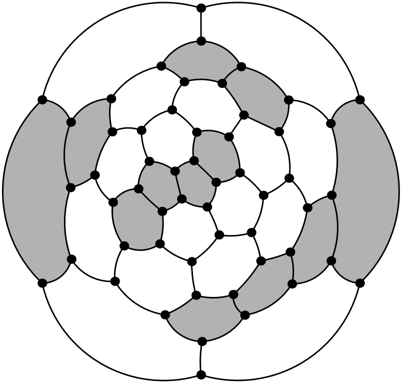

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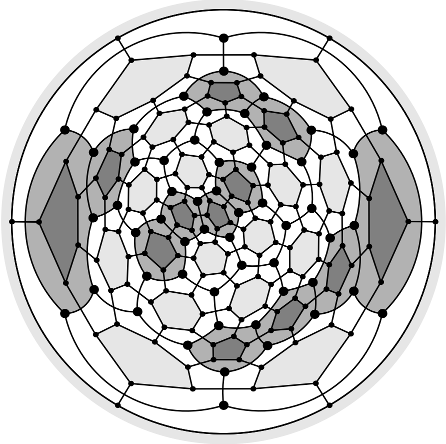

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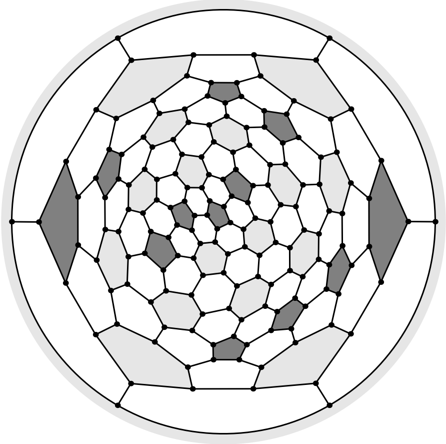

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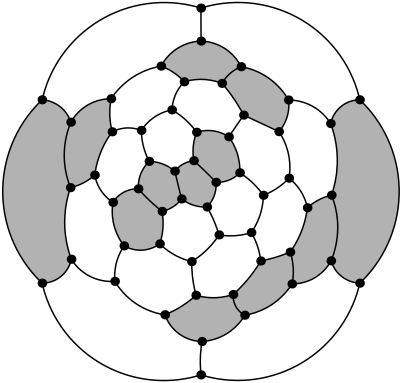

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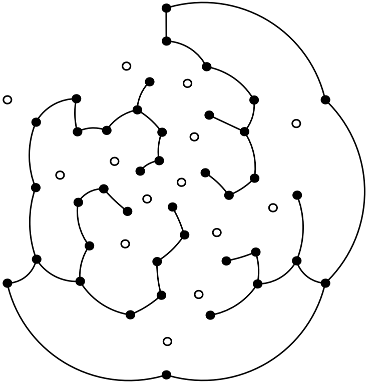

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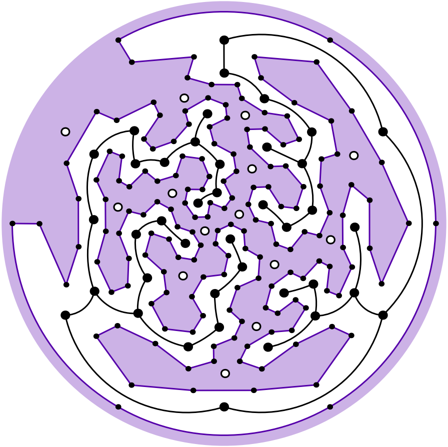

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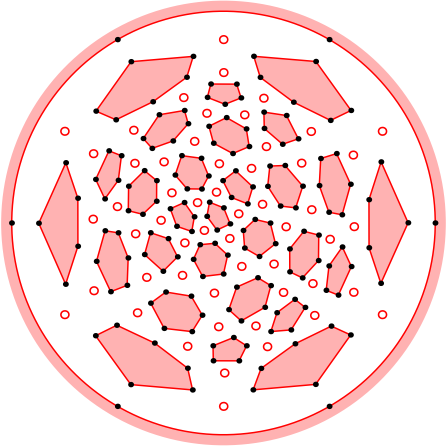

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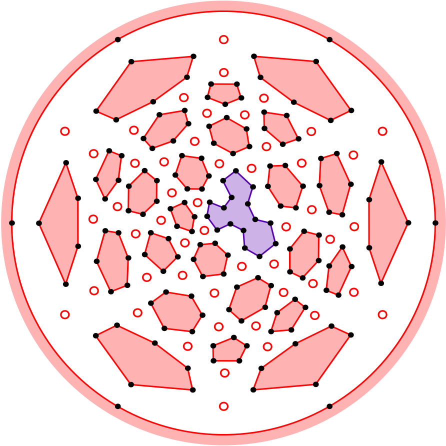

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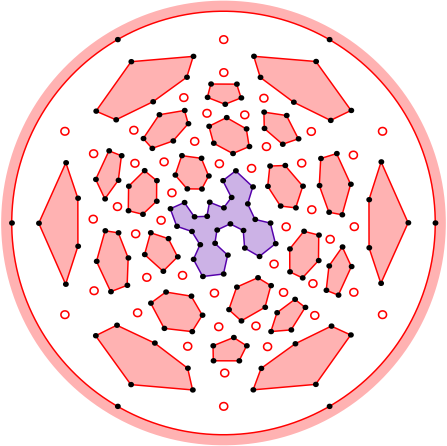

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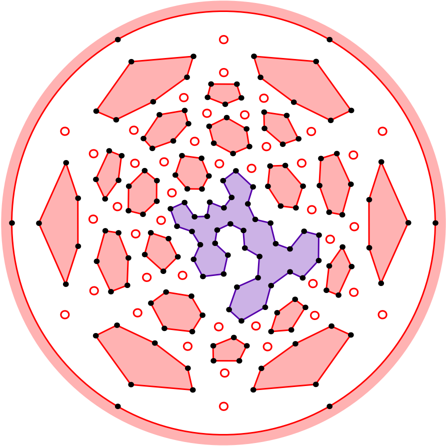

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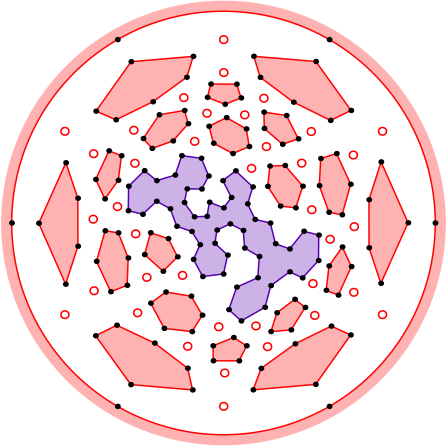

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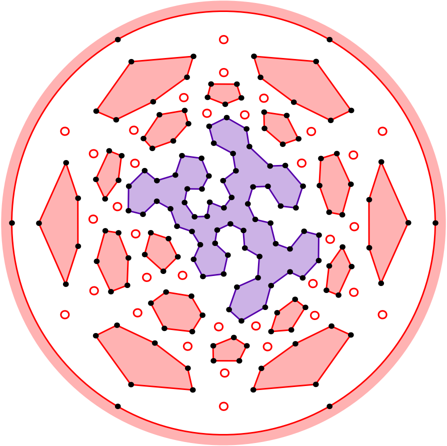

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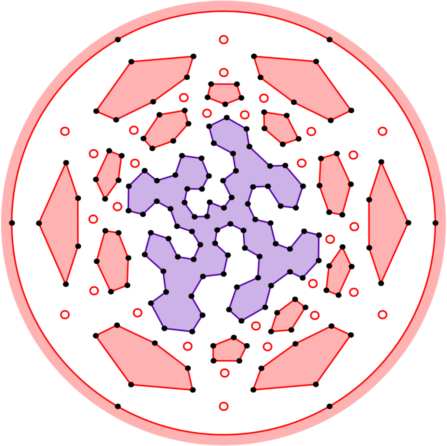

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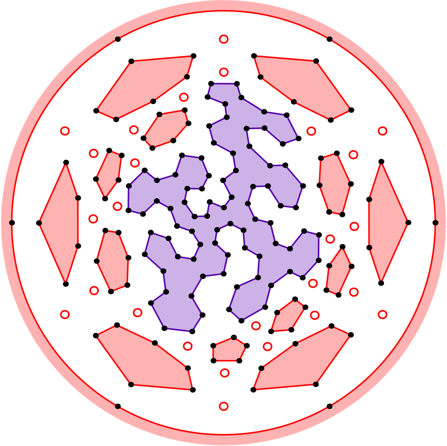

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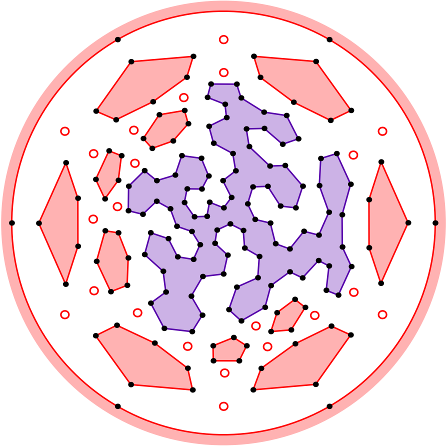

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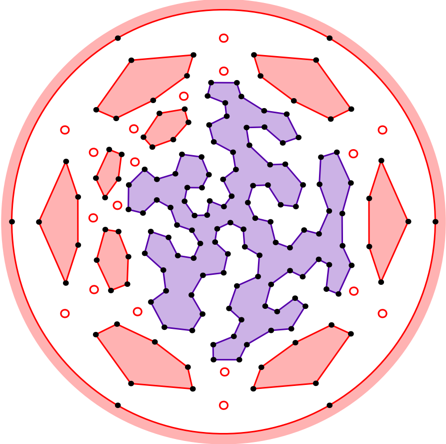

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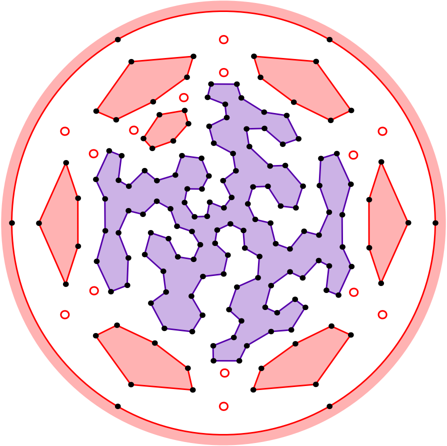

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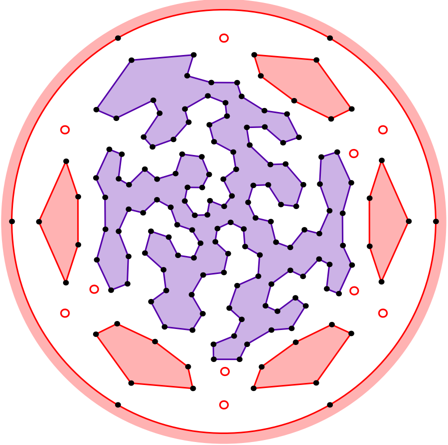

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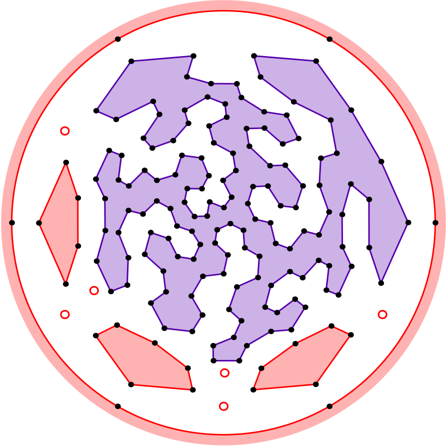

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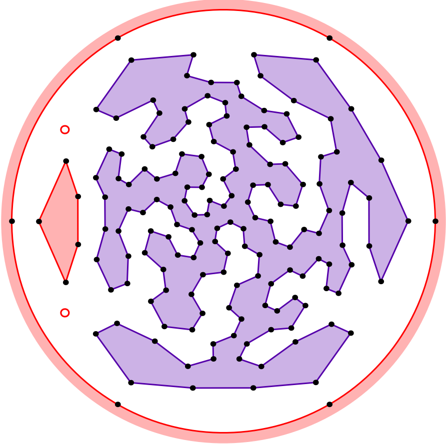

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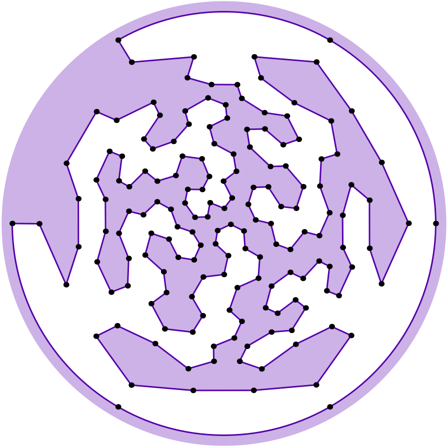

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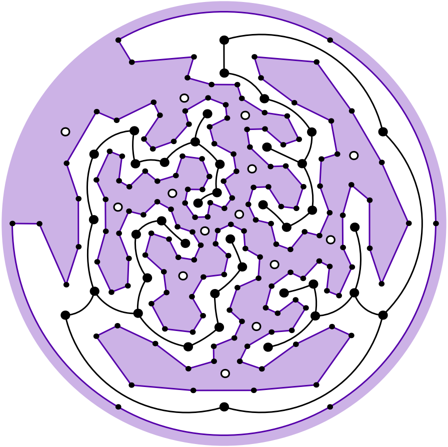

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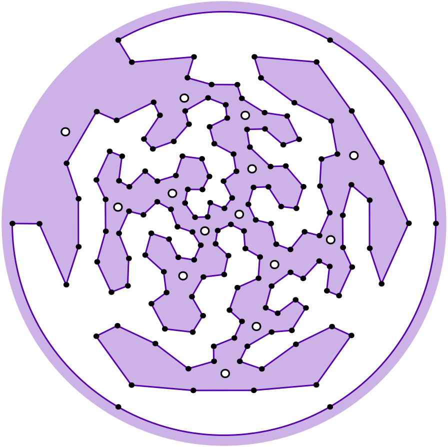

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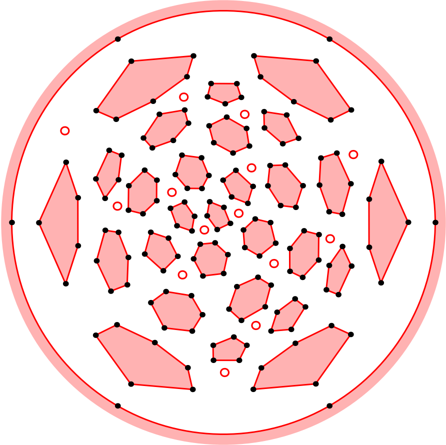

General case

Can we do the same thing for any Barnette graph?

We introduce a linear-time algorithm that finds a Hamilton cycle in a Barnette graph:

- find a 2-factor such that no cycle is inside another
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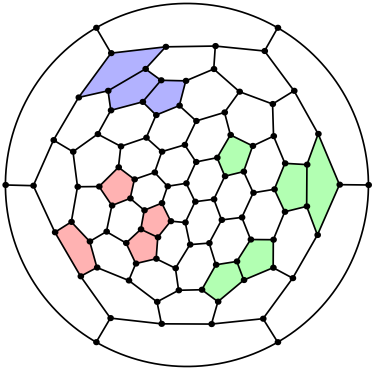
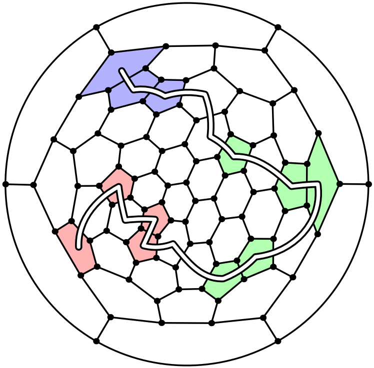
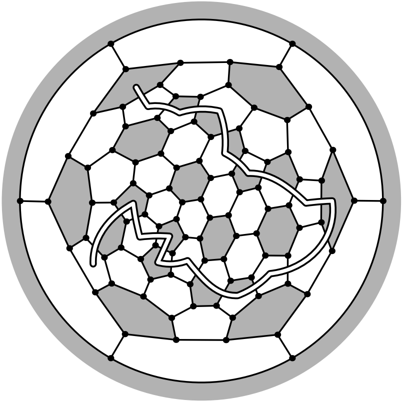
Identify the clusters of pentagons.



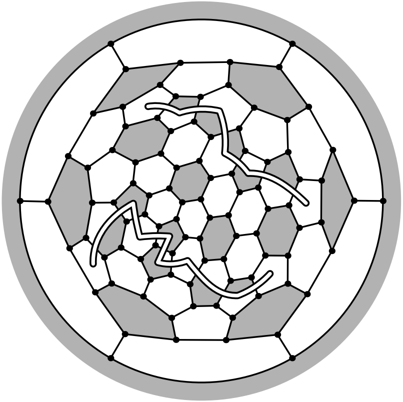
Find a path that meets all the pentagons and cut the graph.



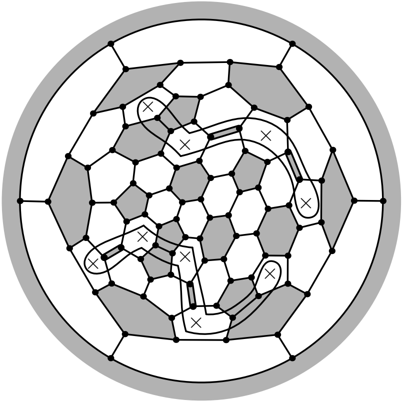
Choose one of the three colors.



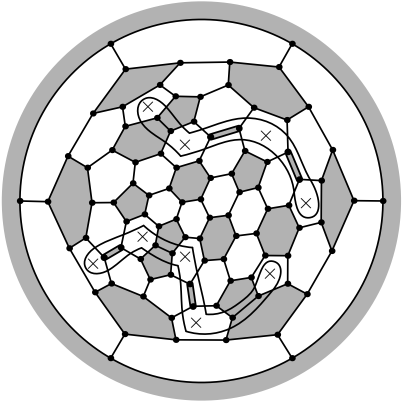
Wherever the two colorings are not compatible,



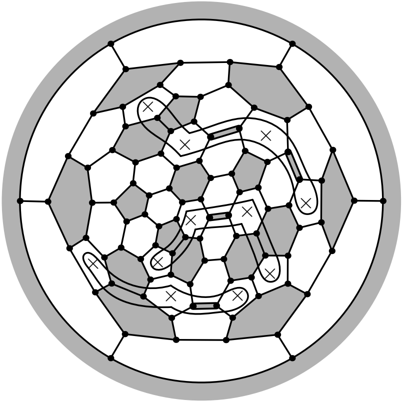
introduce non-resonant white faces.



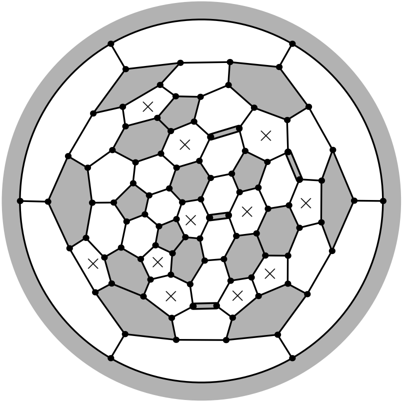
To change the parity of the number of cycles locally,



use a pair of pentagons.



Introduce a new vertex for each resonant hexagon.



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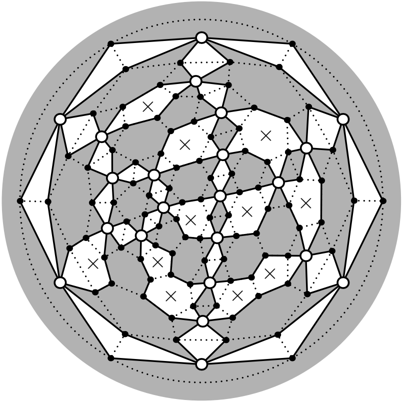
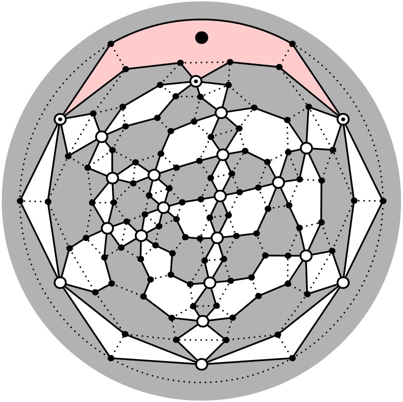
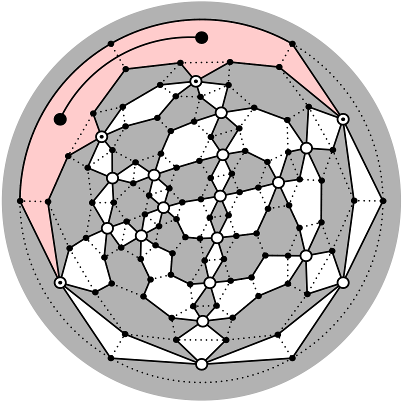


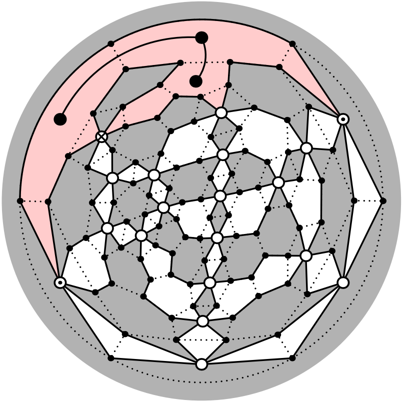
Choose a resonant hexagon to start with and color it red.



Propagate the red color if possible,



if not, glue three black cycles together.



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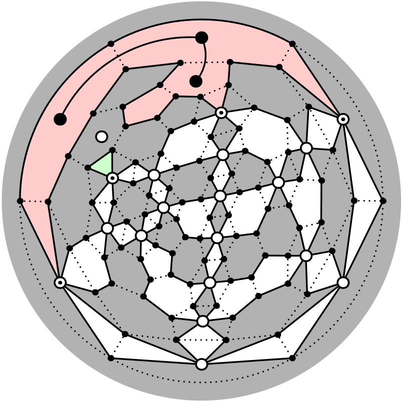

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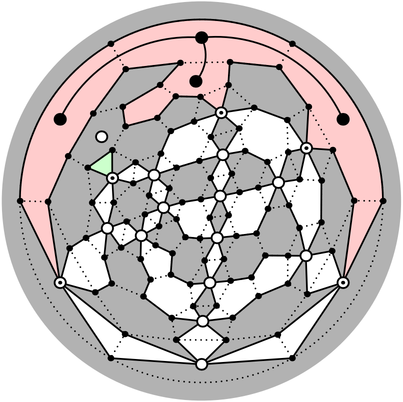

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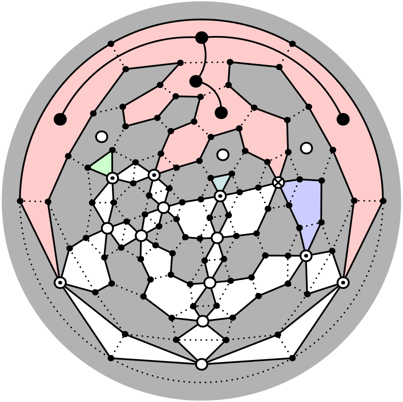

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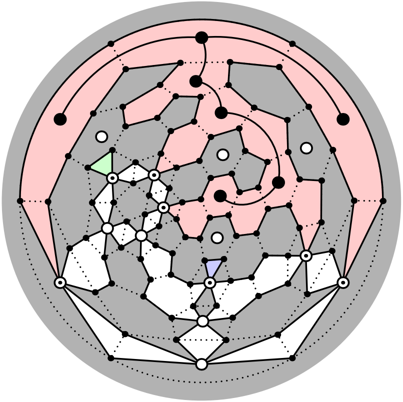

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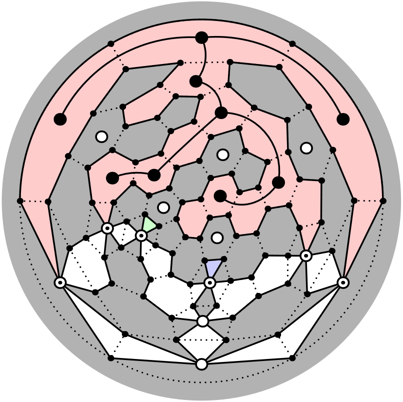

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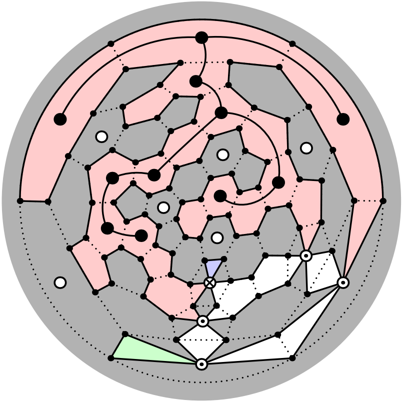
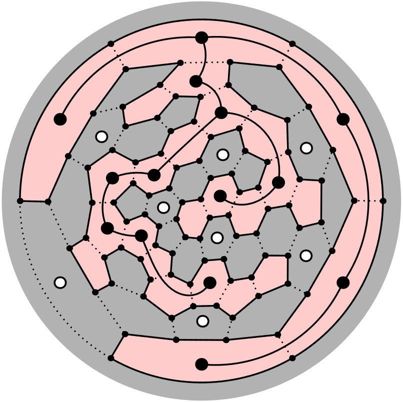


...until there is a single red and a single black face.



In fact, triangles or pairs of adjacent quadrangles are harmless: we can reduce them easily.

Moreover, quadrangles are helpful! We can use a face of size 4 to change the parity of the number of cycles in a 2^*-factor.

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Moreover, quadrangles are helpful! We can use a face of size 4 to change the parity of the number of cycles in a 2^*-factor.

To prove the Barnette's conjecture, it suffices now to examine all the possible configurations of small faces (quadrangles and pentagons) at distance at most two and verify that the algorithm works locally (computer assisted part).

The numbers of clusters containing f_4 quadrangles and f_5 pentagons to be processed:

$f_4 \backslash f_5$	1	2	3	4	5	6	7	8	9
0	1	3	12	92	948	19829	41041	9442	3101
1	3	24	259	5391	1065	297			
2	18	183	389						

Thank you for your attention!