Long properly colored cycles in edge-colored complete graphs

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Joint work with

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- $\Delta^{mon}(G)$: the maximum monochromatic degree of G, i.e,

$$\Delta^{mon}(G)=\max\{\Delta(G^i): i\in col(G)\}$$

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Directed cycles and PC cycles

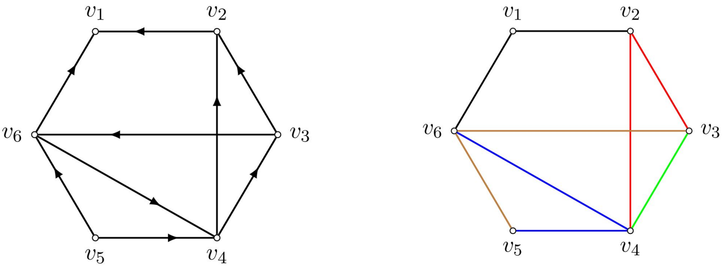

Figure: Transition.

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If $\delta^c(K_n) \geq \frac{n+1}{2}$, then each vertex of K_n is contained in a PC cycle of length k for all $3 \leq k \leq n$.

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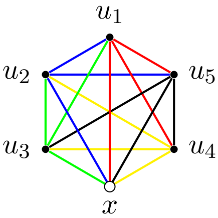

Figure: An example when $\delta^c(K_6) = 3$

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If $\delta^c(K_n) \geq \frac{n+1}{2}$. Then each vertex of K_n is contained in a PC C_3 and a PC C_4. If $n \geq 13$, then each vertex of K_n is contained in a PC cycle of length at least 5.

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- By Theorem 1, $f(n) \geq 4$, and $f(n) \geq 5$ when $n \geq 13$.

Theorem 2 (Li, Broersma, Xu and Zhang, 2016+)

If $\delta^c(K_n) \geq \frac{n+1}{2}$, then each vertex of K_n is contained in a PC cycle of length at least $\delta^c(K_n)$, i.e., $f(n) \geq \delta^c(K_n)$.

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Theorem 3 (Li, Broersma, Xu and Zhang, 2016+)

Let G *be a colored* K_n, *and let* C *be a PC cycle of length* k *in* G. *If* $\delta^c(G) \geq \max\{\frac{n-k}{2}, k\} + 1$, *then* G *contains a PC cycle* C^* *such that* $V(C) \subset V(C^*)$ *and* $|C^*| > k$.

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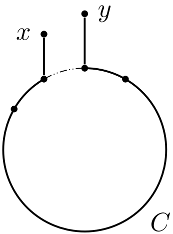

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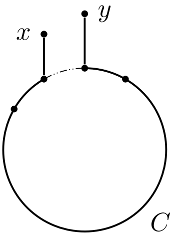
$x \rightarrow C$: $\exists u \in V(C)$ s.t. $col(xu) \notin \{col(uu^+), col(uu^-)\}$.



Proof: By contradiction. Assume that G is a colored K_n with $\delta^c(G) \geq \max\{\frac{n-k}{2}, k\} + 1$ and C is a PC cycle of length k in G. Suppose that there are no PC cycles longer than and containing C. Let $S = V(G) \backslash V(C)$.

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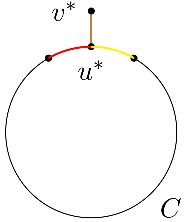
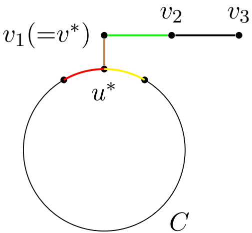
Claim 1

There exist vertices $u^* \in V(C)$ and $v^* \in S$ such that $col(u^*v^*) \notin col(C)$.



Claim 2

For each vertex $v \in S$, there exists a vertex $v' \in S$ such that $col(vv') \notin col(v, C)$.



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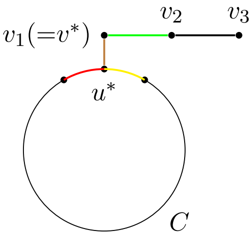

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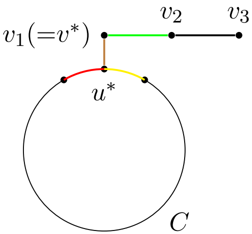

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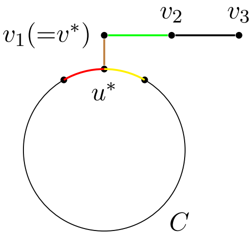


- Lemma 1 forces that $col(v_1v_2) \in col(v_2,C)$
- Lemma 1 and Claim 2 force that $v_1v_2v_3$ is a PC path
- Repeat using Claim 2 and Lemma 1

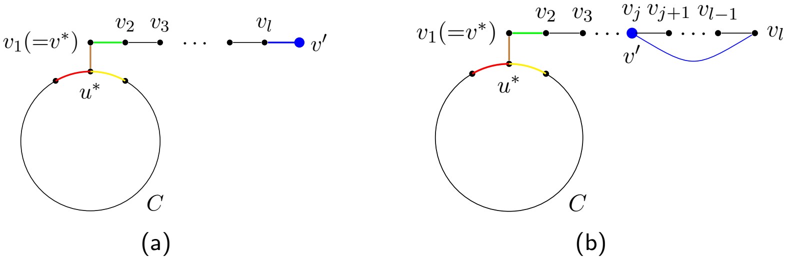
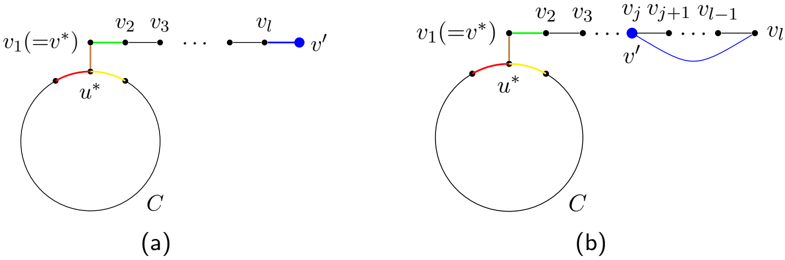


Figure: Two situations when adding a new vertex v'



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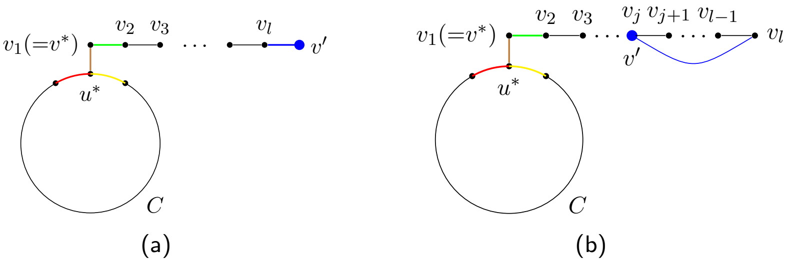

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If case (a) happens, then continue;
Otherwise, set $s = l$; output: $u^*, v_1, v_2, \ldots, v_s$.

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(a) $v_s = v_1$.
(b) $col(v_i, C) = \{col(v_{i-1}v_i)\}$ *for all* i *with* $1 \le i \le s-1$.

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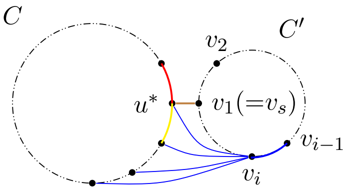

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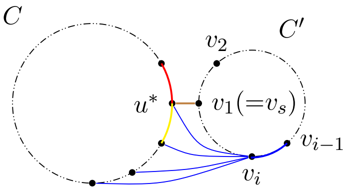

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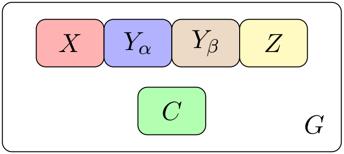

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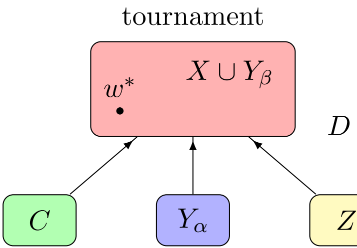

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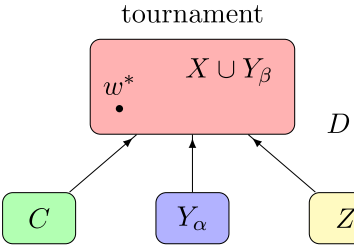
(b) $col(uv) = col(u)$ *for all vertices* $u \in X \cup Y_\beta$ *and* $v \in V(C) \cup Z \cup Y_\alpha$.



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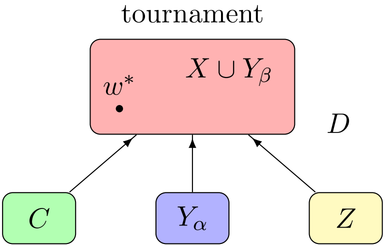
(b) $col(uv) = col(u)$ *for all vertices* $u \in X \cup Y_\beta$ *and* $v \in V(C) \cup Z \cup Y_\alpha$.



$d_D^-(w^*) \geq \frac{|X|+|Y_\beta|-1}{2}$

$col(w^*)$ appears at least $\frac{|X|+|Y_\beta|-1}{2} + |C| + |Y_\alpha| + |Z|$ times at w^*.

Since $d(w^*) = n-1$, we deduce

$$n-1 \geq d^c(w^*) + \frac{|X|+|Y_\beta|-1}{2} + |C| + |Z| + |Y_\alpha| - 1.$$

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Using $n = |X|+|Y_\alpha|+|Y_\beta|+|C|+|Z|$, this yields

$$d^c(w^*) \leq \frac{|X|+|Y_\beta|+1}{2},$$

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or, equivalently,

$$|X|+|Y_\beta| \geq 2d^c(w^*)-1 \geq 2\delta^c(G)-1 \geq 2(\frac{n-|C|}{2}+1)-1 = n-|C|+1.$$

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This implies that $\underline{|X|+|Y_\beta|+|C| \geq n+1}$, a contradiction.

Remark 1: When $k \geq n/3$, then $\max\{\frac{n-k}{2}, k\} = k$.

Corollary 1 (Li, Broersma, Xu and Zhang, 2016+)

Let C be a PC cycle of length k in a colored K_n. If $k \geq \frac{n}{3}$, then there exists a PC cycle C^* of length at least $\delta^c(K_n)$ such that $V(C) \subseteq V(C^*)$.

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Corollary 1 (Li, Broersma, Xu and Zhang, 2016+)

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Question 1

Let C be a PC cycle in a colored K_n. Does there exist a PC C' with $|C'| \geq \delta^c(K_n)$ and $V(C) \subseteq V(C')$? Here, C' is not necessarily distinct from C.

Remark 2: Weaken the conclusion (no "each vertex").

Question 2

Does every colored K_n contains a PC cycle of length at least $\delta^c(K_n)+1$ when $\delta^c(K_n)\geq 2$?

[2] A. Yeo, A note on alternating cycles in edge-colored graphs, *J. Combin. Theory Ser. B*, **69** (1997) 222–225.

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Question 2

Does every colored K_n contains a PC cycle of length at least $\delta^c(K_n)+1$ when $\delta^c(K_n) \geq 2$?

By a result of Yeo [2], when $\delta^c(K_n) = 2$, the answer is Yes.

Theorem 4 (Yeo, JCTB, 1997)

Let G be an edge-colored graph containing no PC cycles. Then there is a vertex $z \in V(G)$ such that no component of $G - z$ is joint to z with edges of more than one color.

[2] A. Yeo, A note on alternating cycles in edge-colored graphs, *J. Combin. Theory Ser. B*, **69** (1997) 222–225.

By a result of Lo [3], when $\delta^c(K_n) \geq \frac{n+1}{2}$, the answer is Yes.

Theorem 5 (Lo, JGT, 2014)

Every edge-colored graph G with $\delta^c(G) \geq 2$ contains a PC path of length $2\delta^c(G)$ or a PC cycle of length at least $\delta^c(G)+1$.

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How about the case $3 \leq \delta^c(K_n) \leq \frac{n+1}{2}$?

[3] A. Lo, A Dirac type condition for properly coloured paths and cycles, Journal of Graph Theory 76 (2014) 60–87.

Thanks for your attention!