Hamiltonicity of graphs on surfaces

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Joint work with

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Hamiltonicity

G : Hamilton-connected

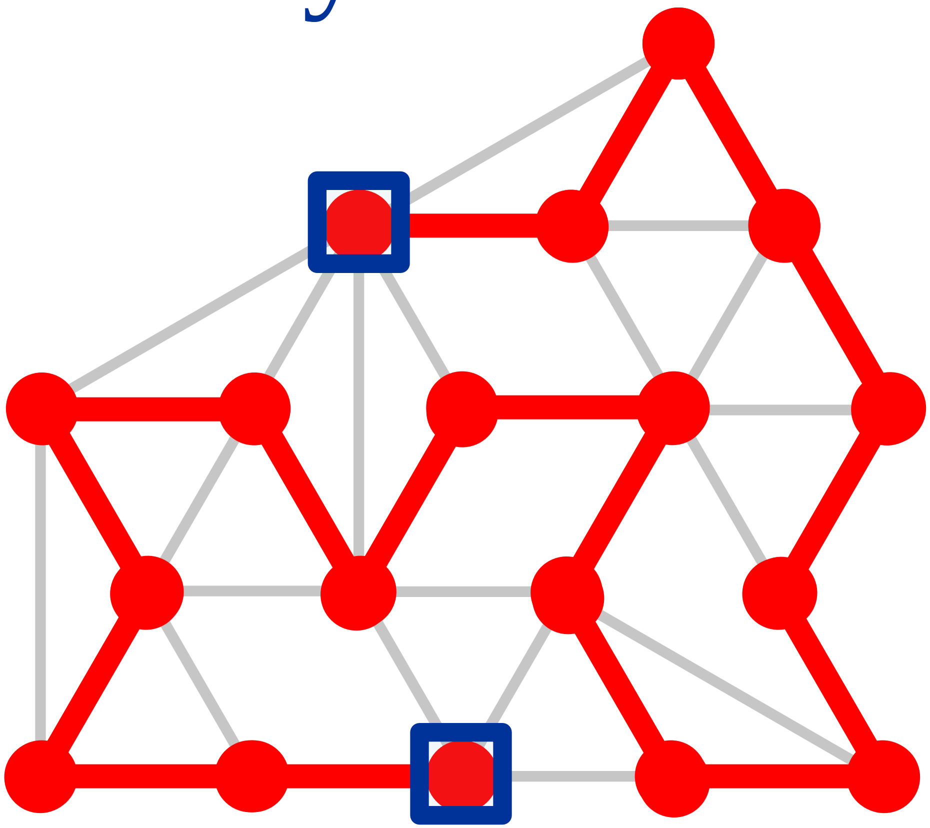
$\Updownarrow$

For $\forall$ pair of vertices,
$\exists$ H-path between them

G : Hamilton-connected

$\Rightarrow$ $\exists$Hamilton cycle

$\Rightarrow$ $\exists$Hamilton path



● $\in V(G)$ ⟋ $\in E(G)$

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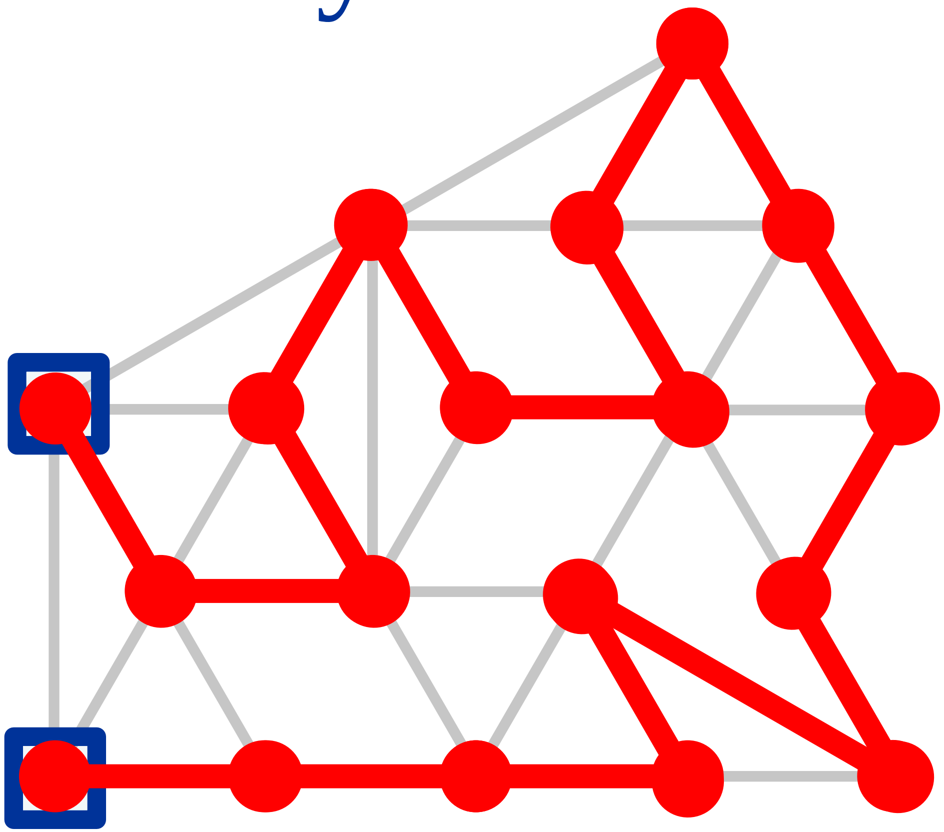
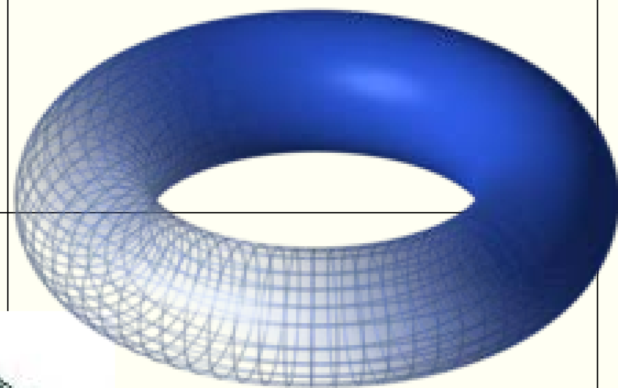
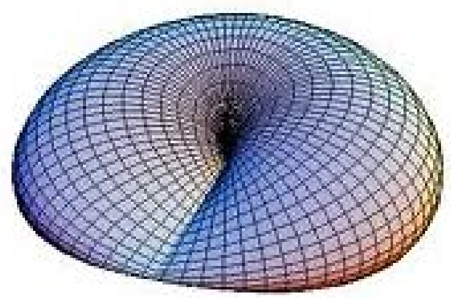


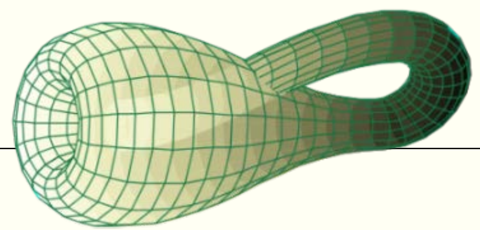
$\bullet \in V(G)$ $\quad$ $\in E(G)$

Hamiltonicity of graphs on surfaces

	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$	K-bottle $\chi = 0$
$\exists$ H-path				
$\exists$ H-cycle				
H-conn				







Hamiltonicity of graphs on surfaces

	4-connected		
	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$
$\exists$ H-path	○	○	○ Thomas, Yu & Zang ('05)
$\exists$ H-cycle	○ Tutte ('56)	○ Thomas & Yu ('94)	? Grunbaum('70) Nash-Williams('73)
H-conn	○ Thomassen ('83)	○ K.K, Oz. (`14+)	✕

Tutte:

$\forall$ 4-conn. plane graph has a H-cycle

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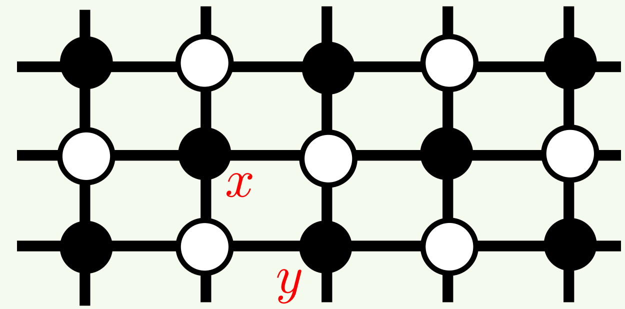
<u>Tutte:</u>

$\forall$ 4-conn. plane graph has a H-cycle

Take a bipartite grid.



$\nexists$ H-path connecting x and y

Hamiltonicity of graphs on surfaces

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H-conn	○ Thomassen (‘83)	○ K.K, Oz. (`14)	×	○ K.K, Oz. (`16)	×	?

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○ Some properties stronger than H-conn.

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C-Tutte path

T : C-Tutte path in G

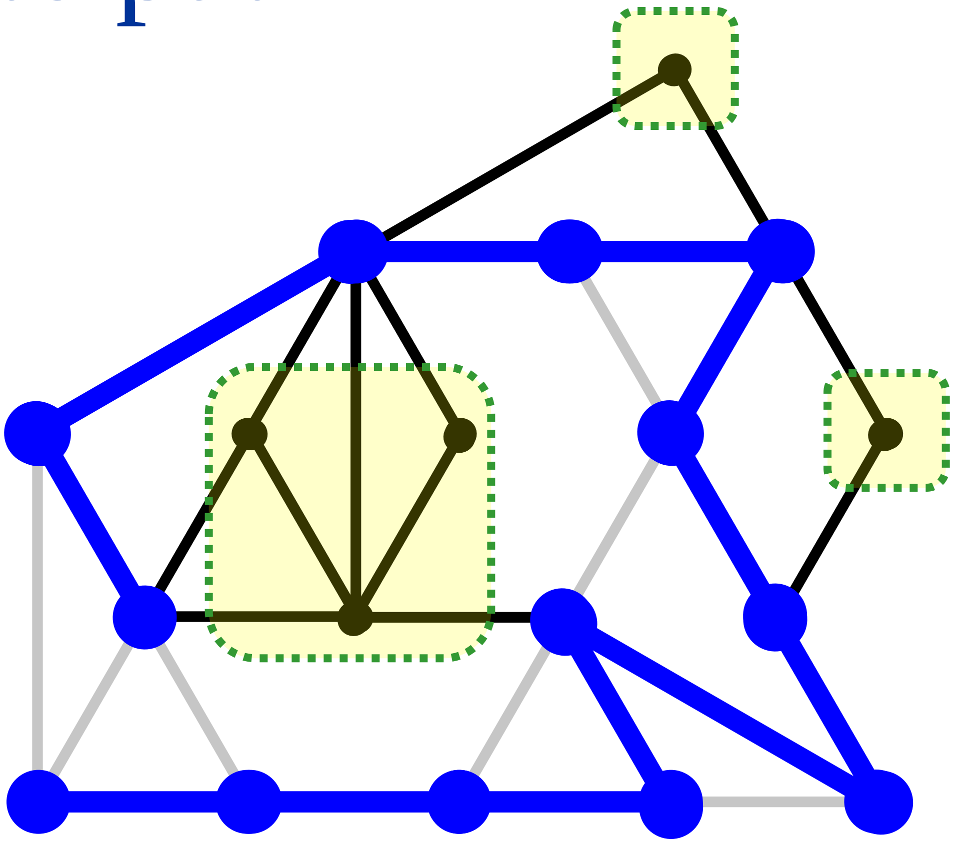
$\Updownarrow$

$\forall B$: compo. of $G - V(T)$

B has ≤ 3 neighbors on T

and ≤ 2 neighbors on T

if B contains a vertex in C



C-Tutte path in G

(C is the outer facial cycle)

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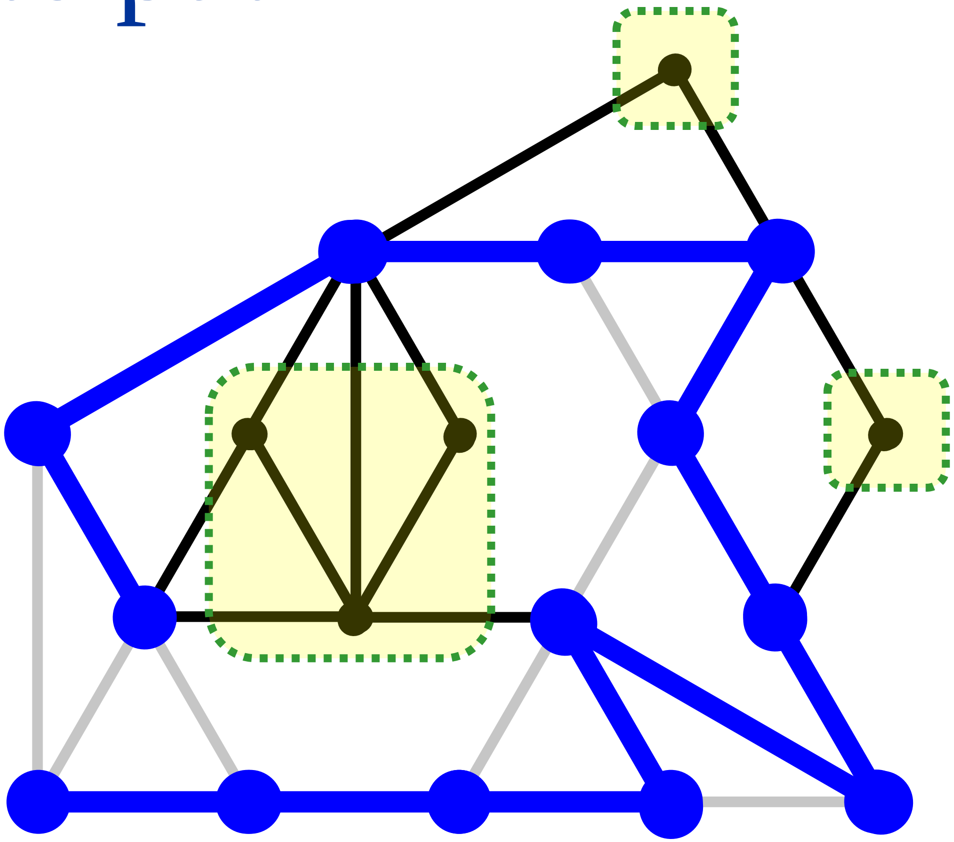
G : 4-conn. $|T| \geq 4$

$\Rightarrow$ T : Hamiltonian path



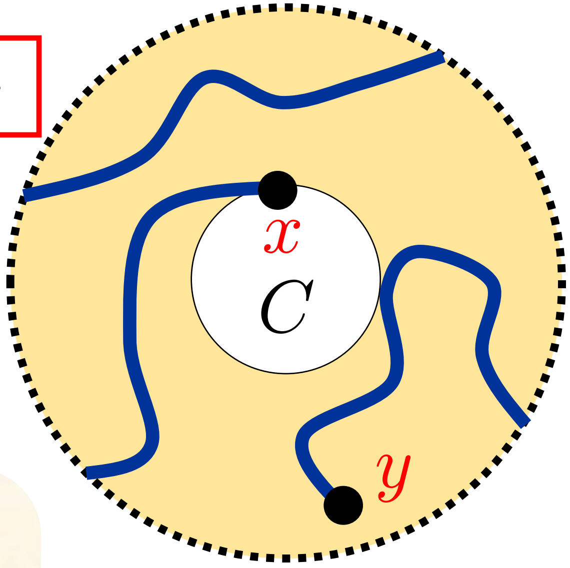
C-Tutte path in G

(C is the outer facial cycle)

Idea of the proof

Condition G : 2-conn.

$x, y \in V(G)$ C : face containing x



Want to find :

T : C-Tutte path between x, y

T : C-Tutte path

$\Leftrightarrow$ For $\forall B$: compo. of $G - V(T)$,

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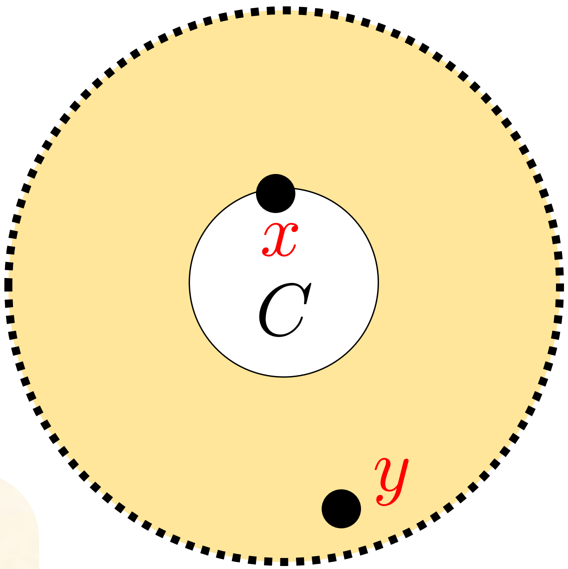
and ≤ 2 neighbors if B contains a vertex in C

Idea of the proof

Ordinary method

Use induction hypothesis to $G - V(C)$



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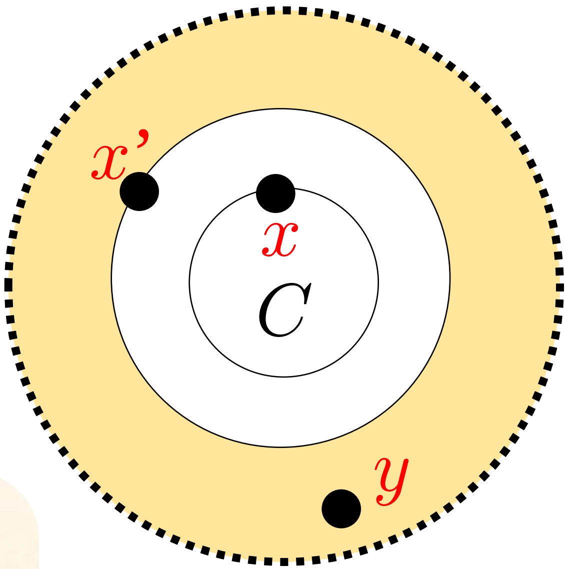

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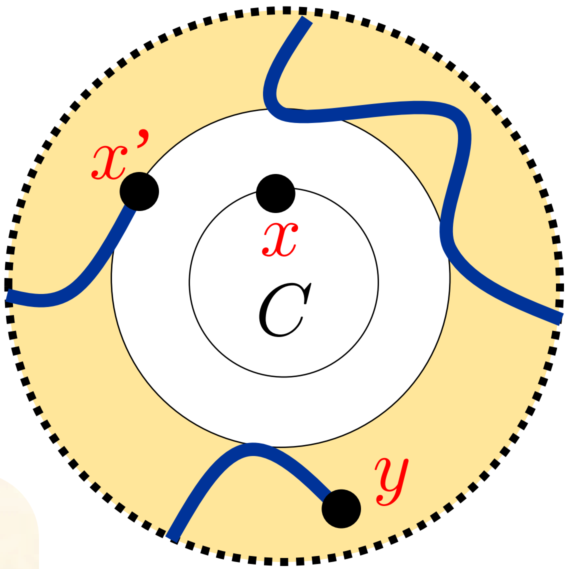
Idea of the proof

<u>Ordinary method</u>

Use induction hypothesis to $G - V(C)$



T : C-Tutte path

$\Leftrightarrow$ For $\forall B$: compo. of $G - V(T)$,

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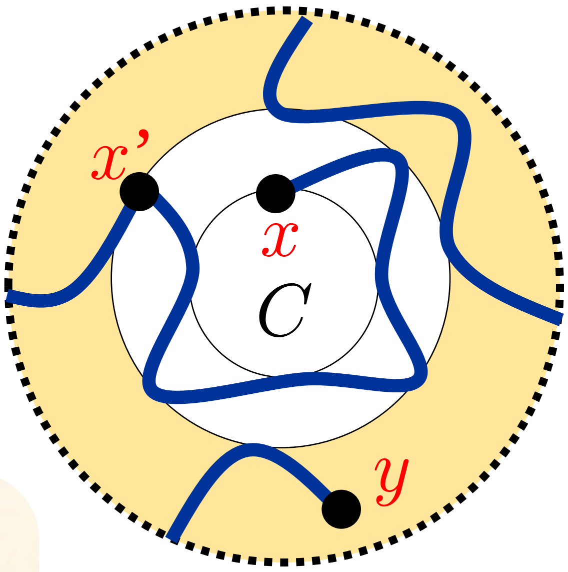
and ≤ 2 neighbors if B contains a vertex in C

Idea of the proof

<u>Ordinary method</u>

Use induction hypothesis to $G - V(C)$ and extend a C-Tutte path



T : C-Tutte path

$\Leftrightarrow$ For $\forall B$: compo. of $G - V(T)$,

B has $\boxed{\leq 3}$ neighbors on T

and $\boxed{\leq 2}$ neighbors if B contains a vertex in C

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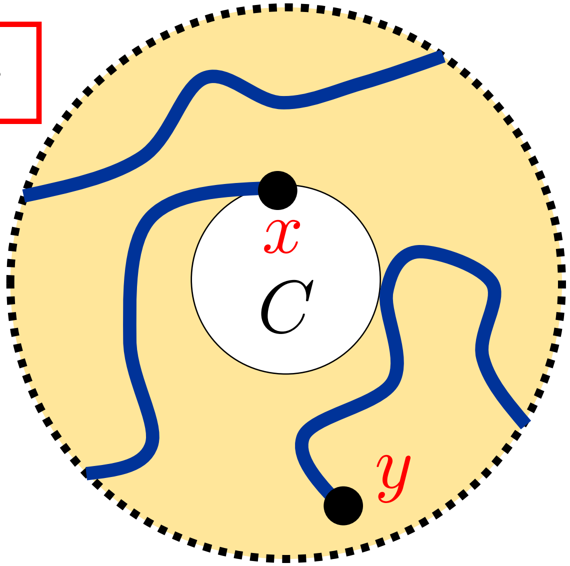
Or



Idea of the proof

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Want to find :

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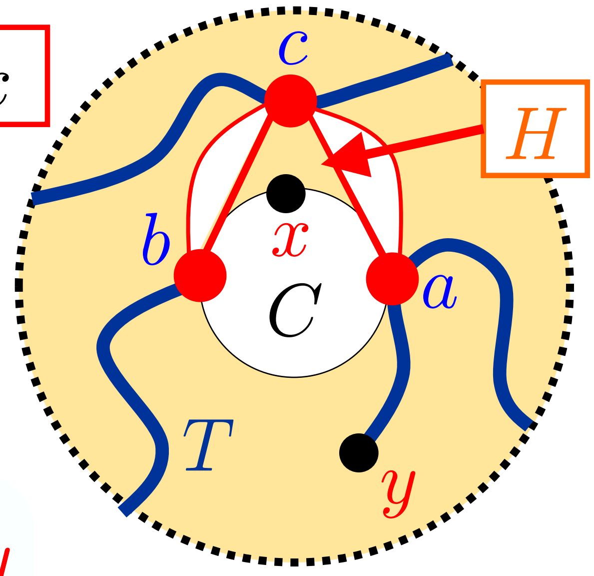
Or

$\exists$ $\{a, b, c\}$: 3-cut of G separating x and y

$\exists$ H : plane comp. of $G - \{a, b, c\}$ containing x (or $x = b$)

$\exists$ T : C-Tutte path in $G - (H - \{a, b, c\})$ between b, y thr. a, c



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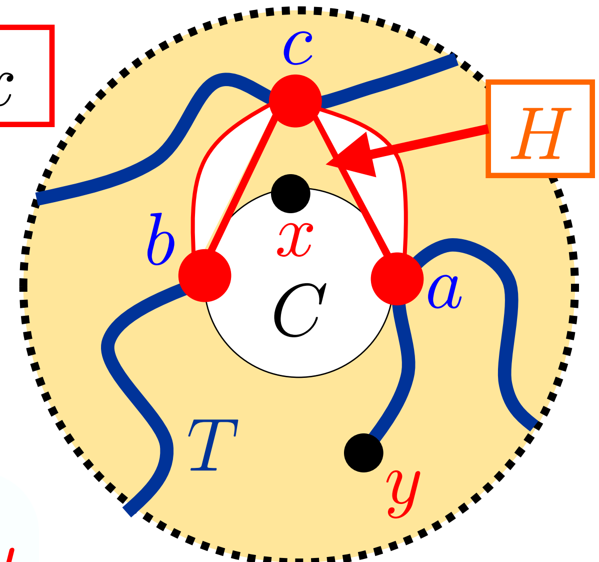

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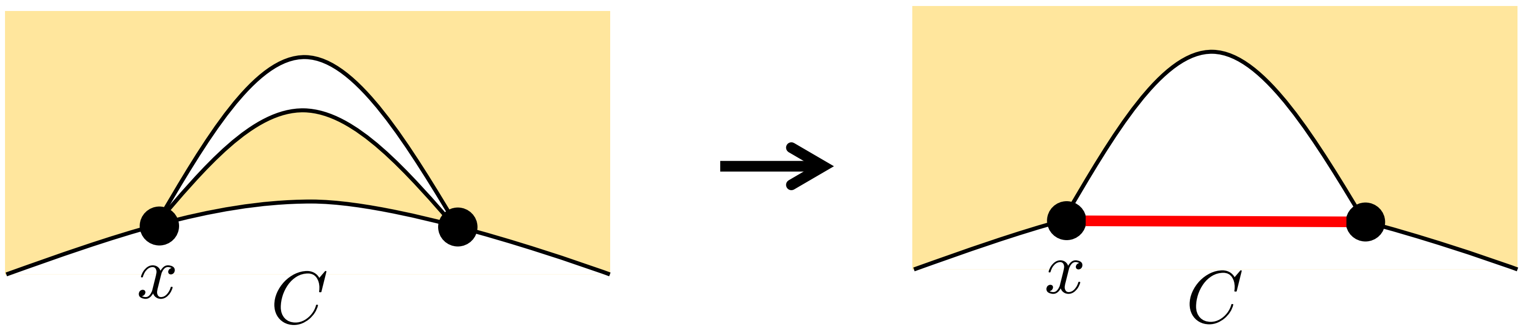
We allow H : (unique) exceptional comp.

Idea of the proof

Induction on $|V(G)|$

Case I : $\exists S$: 2-cut with $x \in S$

We use the reduction appearing in several situations.

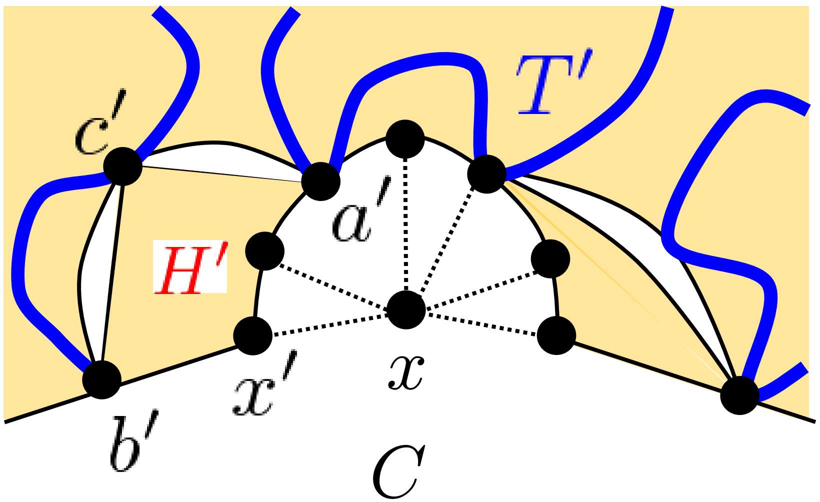


Idea of the proof

Case II : $\nexists S$: 2-cut with $x \in S$

Then $G - x$ is also 2-connected,

and use induction hypothesis to $G - x$



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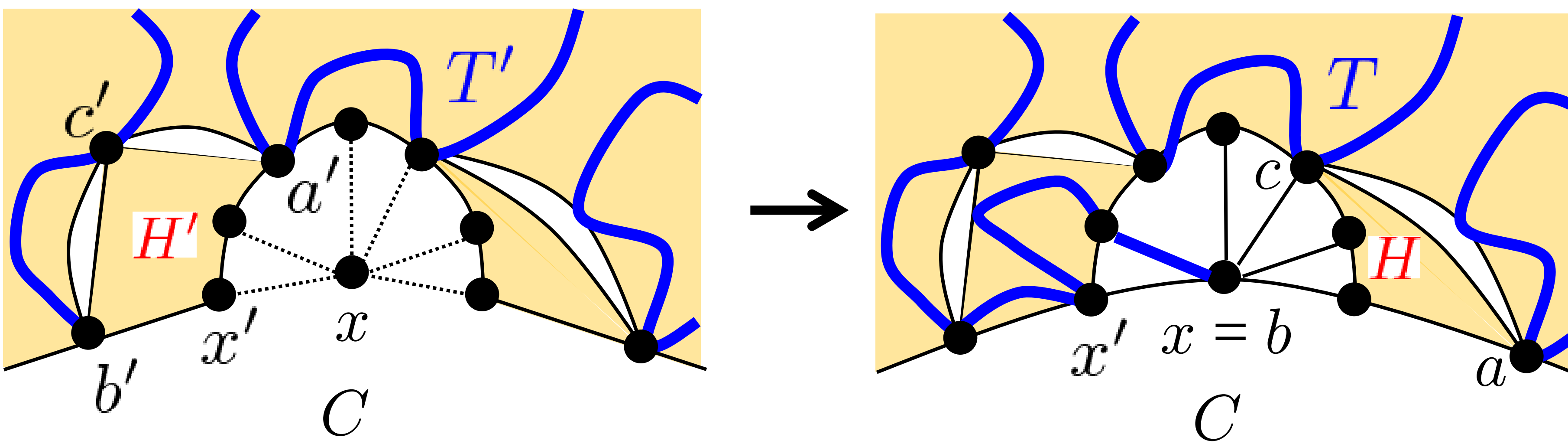


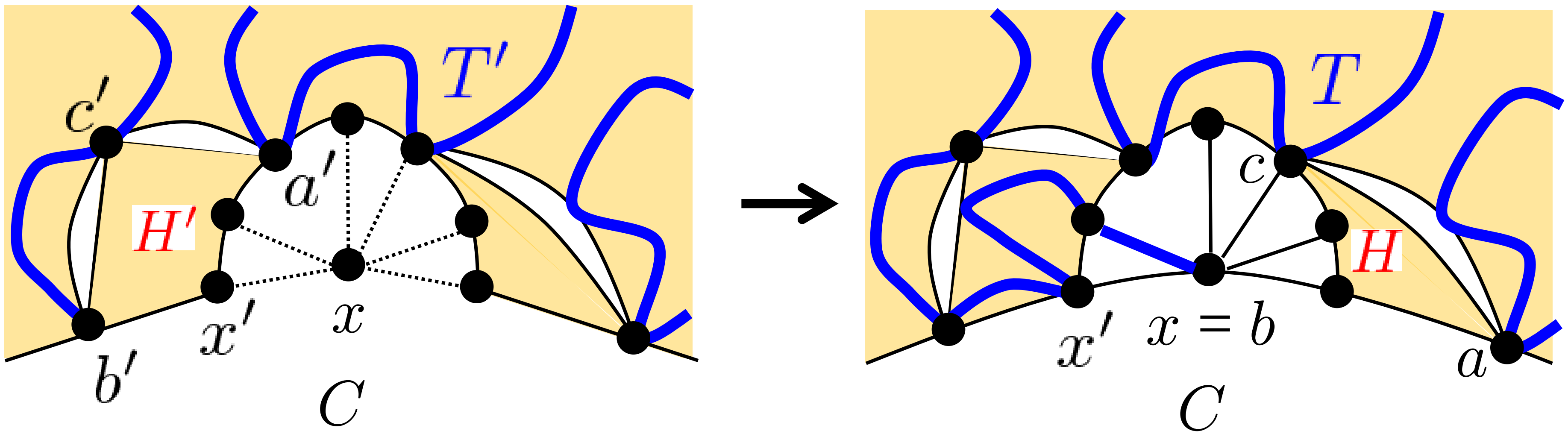
Idea of the proof

Case II : $\nexists S$: 2-cut with $x \in S$

Delete just one vertex

Then $G - x$ is also 2-connected,

and use induction hypothesis to $G - x$



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Thank you for your attention