Flows and edge-colorings on regular graphs

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Regular graphs

We consider (multi-) graphs, i.e. multiedges are allowed (no loops). For an edge $v, w \in V(G)$,

$$\mu(v, w) = \# \text{ edges between } v \text{ and } w$$

$$\mu(G) = \max\{\mu(v, w) : v, w \in V(G)\}.$$

A graph G is r-regular, if $d_G(v) = r$, for each $v \in V(G)$.

An r-regular graph is an r-graph if $|\partial(X)| \geq r$ for each odd $X \subseteq V(G)$.

Edge colorings

A k-edge-coloring of G is a function $\phi : E(G) \longrightarrow \{1, \ldots, k\}$ such that $\phi(e) \neq \phi(f)$ for adjacent edges e and f. The chromatic index $\chi'(G)$ is the smallest number k such that there is k-coloring of G.

Vizing's Theorem (1965)

If G is a graph, then $\chi'(G) \leq \Delta(G) + \mu(G)$.

Shannon's Theorem

If G is a graph, then $\chi'(G) \leq \frac{3}{2}\Delta(G)$.

Edge colorings

A graph G is *class 2* if $\chi'(G) > \Delta(G)$ and *class 1* otherwise.

Every r-regular class 1 graph is an r-graph.

A graph G is k-overfull if $|V(G)|$ is odd, $\Delta(G) = k$ and $\frac{|E(G)|}{\lfloor \frac{1}{2}|V(H)| \rfloor} > k$. It is easy to see that G is k-overfull if and only if $2|E(G)| > k(|V(G)| - 1)$.

Conjecture (Seymour 1979)

If G is an r-graph, then $\chi'(G) \leq r + 1$.

Circular flows

An orientation D of G is an assignment of a direction to each edge, and for $v \in V(G)$, $E^-(v)$ is the set of edges of $E(v)$ with head v and $E^+(v)$ is the set of edges with tail v. The oriented graph is denoted by $D(G)$.
A *nowhere-zero r-flow* $(D(G), \phi)$ on G is an orientation D of G together with a function ϕ from the edge set of G into the real numbers such that

(1) $1 \leq |\phi(e)| \leq r - 1$, for all $e \in E(G)$, and

(2) $\sum_{e \in E^+(v)} \phi(e) = \sum_{e \in E^-(v)} \phi(e), \text{ for all } v \in V(G)$.

Circular flow number

$F_c(G) = \inf\{r | \text{G has a nowhere-zero } r\text{-flow}\}$

is the *circular flow number* of G. It is known, that $F_c(G)$ is always a minimum and that it is a rational number.
That is the number we are interested in. It is a notorious difficult problem to determine the circular flow number of a given graph, or - even more difficult -, to characterize graphs with a specific flow number.

Proposition

A graph G is eulerian if and only if $F_c(G) = 2$.

Conjectures

Conjecture [Tutte 1954]

Let G be a graph. If G is a bridgeless graph, then $F_c(G) \leq 5$.

- equivalent to its restriction on cubic graphs

$F_c(P) = 5$.

Conjecture [Tutte 1966]

If G is a bridgeless graph without 3-edge-cuts, then $F_c(G) \leq 3$.

- equivalent to its restriction on 5-regular graphs

$F_c(K_6) = 3$

Flow numbers of graphs

Theorem [Pan, Zhu 2003]

For every $2 \leq r \leq 5$ there is a graph G with $F_c(G) = r$.

Theorem [es 2001]

Let $t \geq 1$ be an integer, and G be a $(2t+1)$-regular graph. If G is bridgeless, then $F_c(G) = 2 + \frac{1}{t}$ or $F_c(G) \geq 2 + \frac{2}{2t-1}$.

Theorem [Schubert, es 2012]

For every integer $t \geq 1$ and every rational number $r \in \{2 + \frac{1}{t}\} \cup [2 + \frac{2}{2t-1}; 5]$, there exists a $(2t+1)$-regular graph G with $F_c(G) = r$.

Characterizations: smallest possible flow number

Theorem [Tutte]

A cubic graph G is bipartite if and only if $F_c(G) = 3$.

Theorem [es 2001]

Let $t \geq 1$ be an integer. A $(2t+1)$-regular graph G is bipartite if and only if $F_c(G) = 2 + \frac{1}{t}$.

Characterizations: 2^{nd} smallest possible flow number (smallest for non-bipartite graphs)

Theorem [Tutte]

- A cubic graph G is 3-edge-colorable if and only if $F_c(G) \leq 4$.
- A non-bipartite cubic graph G is 3-edge-colorable if and only if $F_c(G) = 4$.

Theorem [es 2013]

Let $t \geq 1$ be an integer. A non-bipartite $(2t+1)$-regular graph G has a 1-factor F such that $G - F$ is bipartite if and only if $F_c(G) = 2 + \frac{2}{2t-1}$.

Sketch of a proof

$(\rightarrow)$ define a flow

$(\leftarrow)$ Let $F_c(G) = 2 + \frac{2}{2t-1}$.

Lemma [es 2001]

Let n, k be integers such that $1 \leq k \leq n$. A graph G has a nowhere-zero $(1 + \frac{n}{k})$-flow if and only if G has a nowhere-zero $(1 + \frac{n}{k})$-flow ϕ such that for each $e \in E(G)$ there is an integer m such that $\phi(e) = \frac{m}{k}$.

There is a nowhere-zero $(2 + \frac{2}{2t-1})$-flow ϕ with $\phi(e) \in \{1, 1 + \frac{1}{2t-1}, 1 + \frac{2}{2t-1}\}$.

Show, that the set $F = \{e : \phi(e) = 1 + \frac{1}{2t-1}\}$ is a 1-factor of G.

sketch of a proof

Let $v \in V(G)$ and $|E^+(v)| > |E^-(v)|$. It turns out that

$\sum_{e \in E^+(v)} \phi(e) \leq t + 1 + \frac{1}{2t-1}$, and $|E^+(v)| = t + 1 = |E^-(v)| + 1$.

Now it follows that for each vertex v precisely one edge of $E(v)$ has flow value $1 + \frac{1}{2t-1}$.

Furthermore, let $v \in V(G)$ and color v black, if $E^+(v) = t + 1$ and white otherwise.
Show that this is a 2-coloring of $V(G)$ which is proper on $G - F$.

Flow numbers and edge-colorings

Corollary

Let $t \geq 1$ be an integer. A $(2t+1)$-regular graph G has a nowhere-zero $(2 + \frac{2}{2t-1})$-flow if and only if G has a 1-factor F such that $G - F$ is bipartite.

Corollary

Let $t \geq 1$ be an integer and G be $(2t+1)$-regular graph. If $F_c(G) \leq 2 + \frac{2}{2t-1}$, then G is a class 1 graph.

Observation

There is a flow number, namely 4, that separates cubic class 1 and cubic class 2 graphs from each other.

Does there exist such number for $(2t+1)$-regular graphs if $t > 1$.

No!

We think that such a number exists for cubic graphs is just due to the fact that $\frac{2}{2t-1} = \frac{2}{t}$ iff $t = 1$.

Proposition [es 2001]

For $k \geq 1$ holds: $F_c(K_{2k+2}) = 2 + \frac{2}{k}$.

The interval $(2+\frac{2}{2t-1}, 2+\frac{2}{t})$

Proposition

For every integer $t > 1$ and every rational number $r \in \{2+\frac{1}{t-1}\} \cup [2+\frac{2}{2t-3}; 5]$, there exists a $(2t+1)$-regular class 2 graph G with $F_c(G) = r$.

Proposition

For every integer $t > 1$ there are $(2t+1)$-regular graphs G_1 and G_2 such that G_1 is a class 1 graph, G_2 is a class 2 graph, and $F(G_1) = F(G_2) = 2+\frac{2}{t}$.

The interval $(2+\frac{2}{2t-1}, 2+\frac{2}{t})$

Every k-regular class 1 graph is a k-graph.

Theorem

For every integer $t \geq 1$ there is a $(2t+1)$-graph G which is a class 2 graph and $F_c(G) = 2 + \frac{3}{3t-2}$.

If $t > 1$, then $\frac{2}{2t-1} < \frac{3}{3t-2} < \frac{2}{t}$.

Problems and Conjectures

For an integer $t \geq 1$ let

$$\Phi(2t+1) = \inf\{F_c(G) : G \text{ is a } (2t+1)\text{-regular class 2 graph}\}.$$

Conjecture

For every integer $t \geq 1$: $\Phi(2t+1) = 2 + \frac{2}{2t-1}$.

Problem

Is it true that for every integer $t > 1$ and every rational number r with $2 + \frac{2}{2t-1} < r \leq 2 + \frac{2}{t}$ there are $(2t+1)$-regular graphs H_1 and H_2 such that H_1 is class 1, H_2 is class 2, and $F_c(H_1) = F_c(H_2) = r$.

Problems and Conjectures

Conjecture

Let $t \geq 1$ be an integer and G a $(2t+1)$-regular graph. If G is a class 1 graph, then $F_c(G) \leq 2 + \frac{2}{t}$.

For $t = 1$ true.

Conjecture

Let $t > 1$ be an integer. If G is a $(2t+1)$-graph, then $F_c(G) \leq 2 + \frac{2}{t}$.

This conjecture says that every $(2t+1)$-regular graph G with $F_c(G) > 2 + \frac{2}{t}$ has an overfull subgraph H with $\Delta(H) = 2t+1$.

For $t = 1$ true. For $t = 2$ it is Tutte's 3-flow conjecture.
If t is even, say $t = 2t'$, then it is Jaeger's circular flow conjecture for $(4t'+1)$-regular graphs. Jaeger conjectured that every $4t'$-connected graph has a $(2 + \frac{1}{t'})$-flow.

Thank you very much!