On the minimum degree of minimally 1-tough graphs

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Connectivity

Definition

A graph G is said to be k-connected, if it has at least $k+1$ vertices and remains connected whenever fewer than k vertices are removed. The connectivity of G, denoted by $\kappa(G)$, is the largest k for which G is k-connected.

Definition

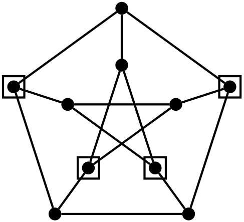
A graph G is said to be minimally k-connected, if $\kappa(G) = k$ and $\kappa(G - e) < k$ for all $e \in E(G)$.

Definition

Let t be a positive real number.
A graph G is called t-tough, if

$$\omega(G - S) \leq \frac{|S|}{t}$$

for any cutset S of G.
The toughness of G, denoted by $\tau(G)$, is the largest t for which G is t-tough, taking $\tau(K_n) = \infty$ for all $n \geq 1$.



The Petersen graph is 4/3-tough.

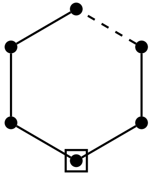
A cycle is 1-tough.

Example (1-tough graphs)

The removal of any k vertices leaves at most k components.

Definition

A graph G is said to be minimally t-tough, if $\tau(G) = t$ and $\tau(G - e) < t$ for all $e \in E(G)$.

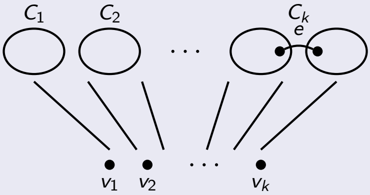


A cycle (of length at least 4) is minimally 1-tough.

Example (Minimally 1-tough graphs)

For every edge e there exists a vertex set S for which

$$\omega(G - S) = |S|,$$
$$\omega((G - e) - S) = |S| + 1.$$



Minimum degrees

Proposition (Chvátal, 1973)

If G is a noncomplete graph, then $\kappa(G) \geq 2\tau(G)$.

Proof.

If G is noncomplete, then there exists a cutset S of size $\kappa(G)$. Then $\omega(G - S) \geq 2$, so

$$\tau(G) \leq \frac{|S|}{\omega(G - S)} \leq \frac{\kappa(G)}{2}.$$

In other words: a t-tough graph is at least $2t$-connected.

Theorem (Mader, 1971)

Every minimally k-connected graph has a vertex of degree k.

Conjecture (Kriesell)

Every minimally 1-tough graph has a vertex of degree 2.

Toughness and hamiltonian cycles

If a graph contains a hamiltonian cycle, then it is 1-tough, but the converse is not necessarily true (Petersen graph).

Proposition

Every minimally 1-tough hamiltonian graph is a cycle.

Proof. Otherwise, remove an edge not contained by the hamiltonian cycle. The remaining graph is still hamiltonian, so it is 1-tough.

Conjecture (Chvátal)

There exists a constant t_0 such that every t_0-tough graph is hamiltonian.

Theorem (Bauer, Broersma, Veldman, 2000)

For every $\varepsilon > 0$ there exists a $(9/4 - \varepsilon)$-tough graph having no hamiltonian path.

Theorem (G.Y. K., D. S., K. V.)

Every graph can be embedded as an induced subgraph into a minimally t-tough graph.

Proof.

Step 1 Embed the graph as an induced subgraph into an α-critical graph G.

Definition

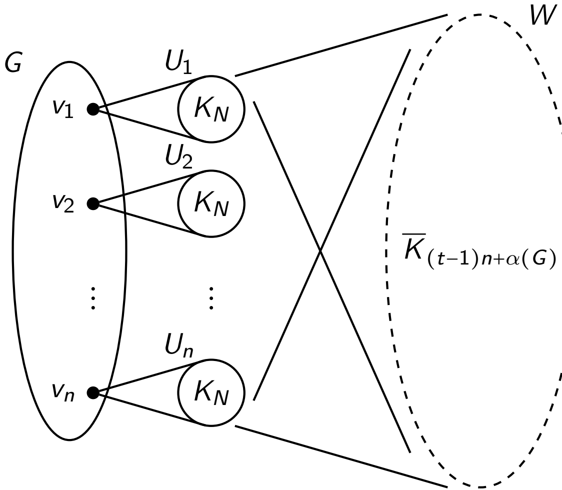
A graph G is called α-critical, if $\alpha(G - e) > \alpha(G)$ for all $e \in E(G)$.

Lemma

Every graph can be embedded as an induced subgraph into an α-critical graph.

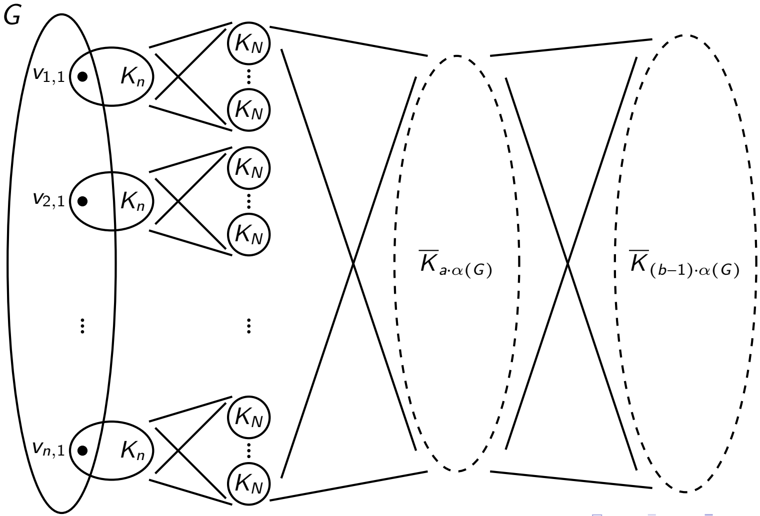
Constructions, $t \geq 1$

Step 2, embedding the α-critical graph into a (not necessarily minimally) t-tough graph, if $t \geq 1$



Constructions, $t < 1$

Step 2, embedding the α-critical graph into a (not necessarily minimally) t-tough graph, if $t < 1$

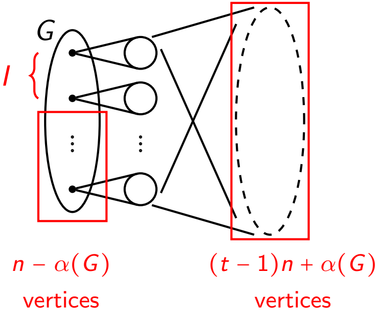


Step 2, both cases, continuing

Claim

$\tau(H) = t$.

In G there exists an independent set I of size $\alpha(G)$.
If $t \geq 1$:



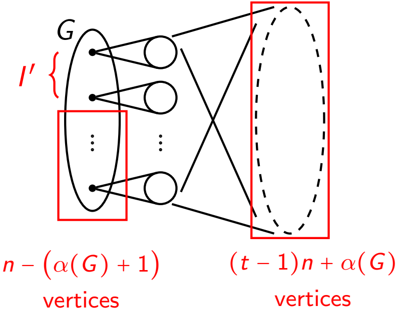
$$\frac{|S|}{\omega(G-S)} = \frac{tn}{n} = t.$$

Theorem

For each edge $e \in E(G)$, $\tau(H - e) < t$.

Proof.

In $G - e$ there exists an independent set I' of size $\alpha(G) + 1$.



Toughness:

$$\frac{|S'|}{\omega(G - S')} = \frac{tn - 1}{n} < t.$$

Step 3 Remove some edges, until the graph remains t-tough.

Conjecture (Kriesell)

Every minimally 1*-tough graph has a vertex of degree* 2*.*

Trivial upper bound

Proposition

Every minimally 1*-tough graph has a vertex of degree* $(n-1)/2$*.*

Proof. If each vertex has degree at least $n/2$, then by Dirac's theorem the graph is hamiltonian, but not a cycle.

Lemma (A useful property of minimally 1-tough graphs)

If G is a minimally 1-tough graph, then for every edge e, there exist $k = k(e)$ vertices, whose removal from G leaves exactly k components and from $G - e$ leaves exactly $k + 1$ components.

Theorem (G.Y. K., K. V.)

Every minimally 1-tough graph has a vertex of degree at most $(n + 2)/3$.

Sketch of proof.

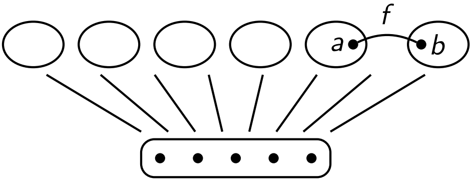
Suppose to the contrary, that G is a minimally 1-tough graph with $\delta(G) \geq n/3 + 1$.

Lemma

For every edge e, $k(e) > n/3$.

Lemma

There exists an edge $f = \{a, b\}$ such that there is at most $n/3 - 1$ vertices which are not adjacent either to a or to b.



$k > n/3$
components

$k > n/3$
vertices

For this we need $k + 1 > n/3 + 1$ independent vertices,
two of them must be a and b,
the rest of them cannot be adjacent either to a or to b. $\lightning$ $\square$

Toughness of claw-free graphs

Definition

The graph $K_{1,3}$ is called a claw. A graph is said to be claw-free, if it does not have a claw as an induced subgraph.

Theorem (Matthews, Sumner, 1984)

If G is a claw-free, noncomplete graph, then $\kappa(G) = 2\tau(G)$.

Corollary

If G is a claw-free, minimally $2t$-connected, noncomplete graph, then it is minimally t-tough.

Proof.

By the Matthews–Sumner Theorem, G is t-tough.
If we remove an arbitrary edge, then the remaining graph is no longer $2t$-connected, so it is no longer t-tough.

Toughness of claw-free graphs

Question (Is the converse true?)

If G is a claw-free, minimally t-tough, noncomplete graph, then is it minimally $2t$-connected?

Difficulty: if we remove an edge from a claw-free graph, then the remaining graph is not necessarily claw-free.

Proposition

It follows easily, that a $\{K_{1,3}, K_{1,3}+e\}$-free noncomplete graph is minimally $2t$-connected iff it is minimally t-tough.

Proof.

$\Rightarrow$: $\checkmark$

$\Leftarrow$: By the Matthews–Sumner Theorem, G is $2t$-connected.
If we remove an arbitrary edge, then the remaining graph is no longer t-tough and claw-free, so by the Matthews–Sumner Theorem it is no longer $2t$-connected.

However, this graph class is not large.

Proposition

If G is a $\{K_{1,3}, K_{1,3}+e\}$-free graph and $\kappa(G) = 2t$, where $t > 1$, then G can be created from the complete graph K_{2t+2} by removing an independent edge set.

Theorem (G.Y. K., K. V.)

If G is a minimally 1-tough claw-free graph of order n, then $G = C_n$.

Corollary

1. *Kriesell's Conjecture is true for claw-free graphs.*
2. *A claw-free noncomplete graph is minimally 2-connected iff it is minimally 1-tough.*

Our results

- Every graph can be embedded as an induced subgraph into a minimally t-tough graph.
- Every minimally 1-tough graph has a vertex of degree at most $(n+2)/3$.
- Every minimally 1-tough claw-free graph is a cycle.

Thank you for your attention!