Computational Results on the α-Deficit of Trees

Gunnar Brinkmann, Hadrien Mélot and Eckhard Steffen

Gunnar.Brinkmann@UGent.be
Hadrien.Melot@umons.ac.be
es@uni-paderborn.de

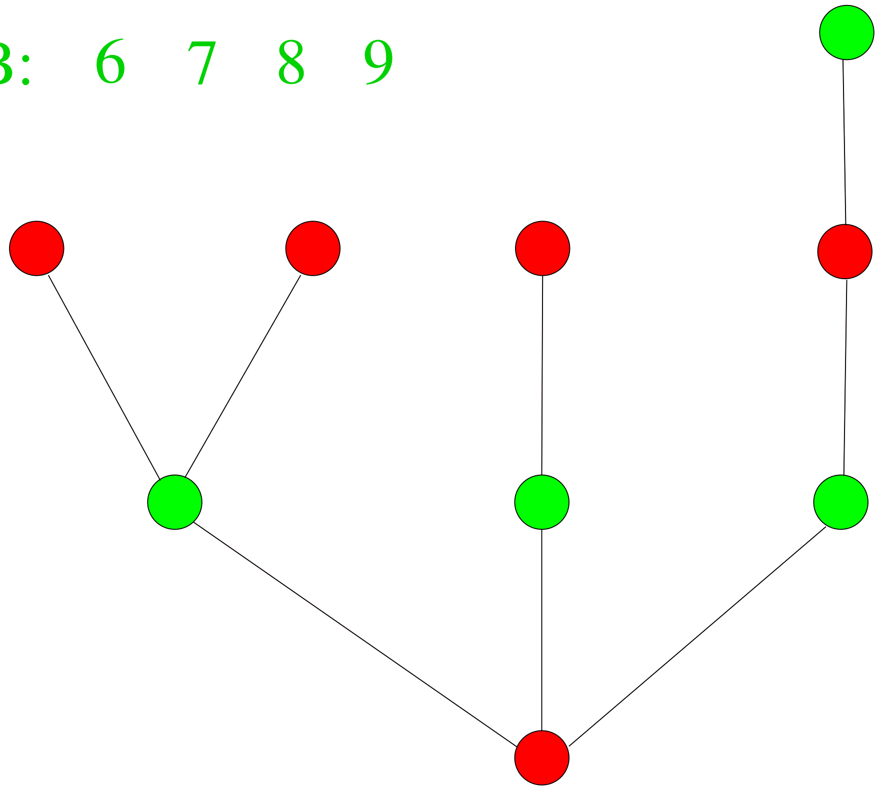
Bipartite labeling of a tree:

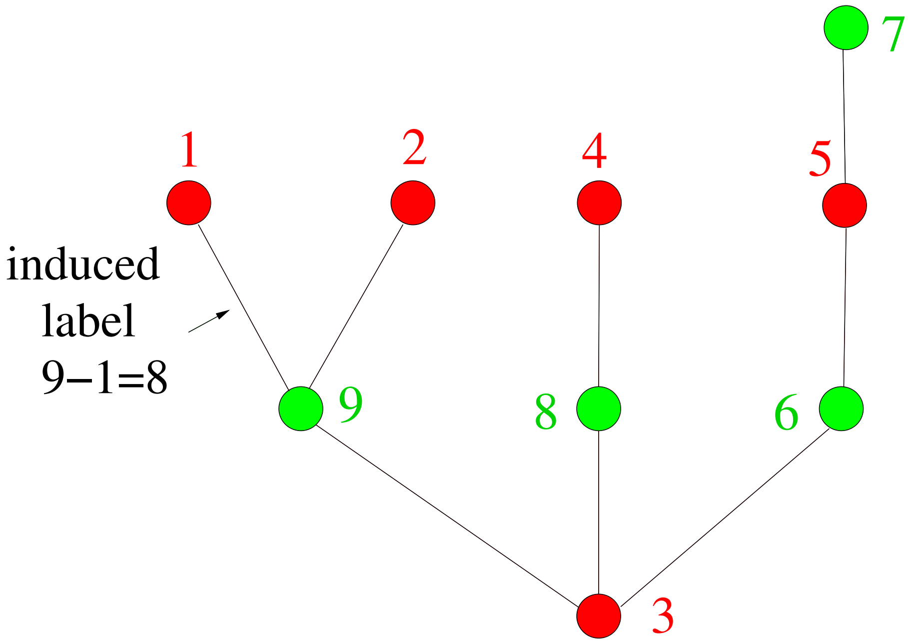
Given a tree with bipartition classes A and B.

Label the vertices in A with $1, \ldots, |A|$ and the vertices in B with $|A|+1, \ldots, |A|+|B|$.

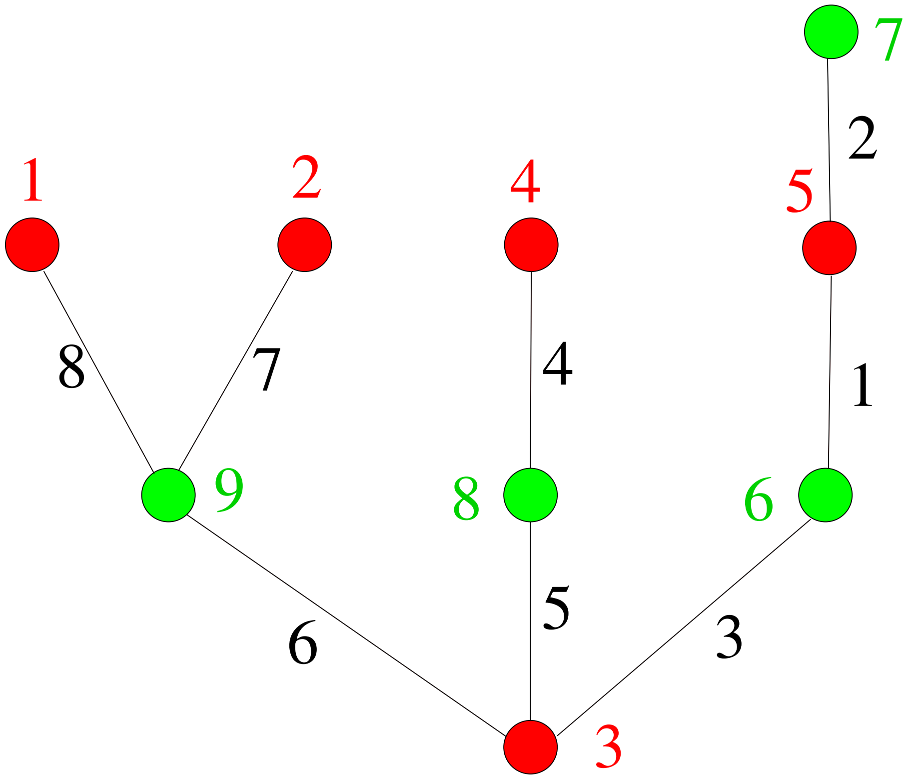
A: 1 2 3 4 5

B: 6 7 8 9




7
1
2
4
5
induced
label
9−1=8
9
8
6
3

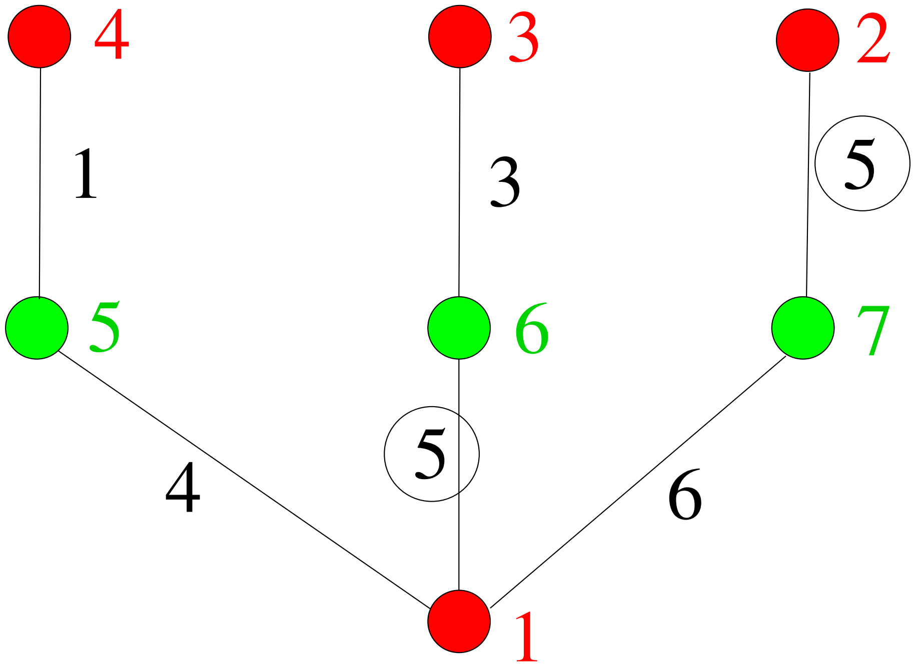
All possible edge labels 1, . . . , 8 are present:



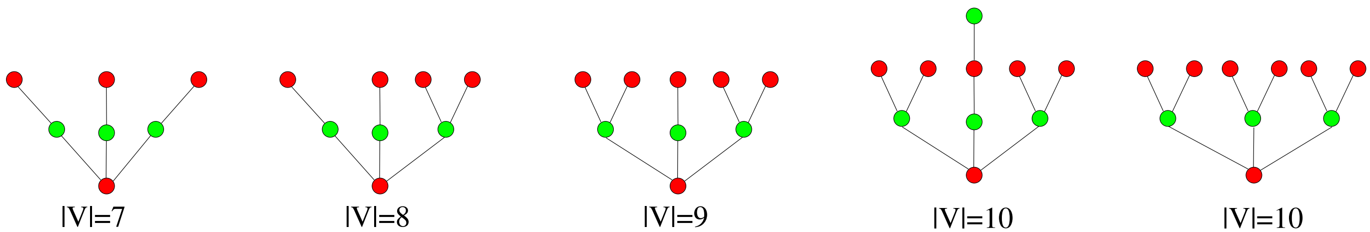
Definition:

The α-deficit of a tree with n vertices is the minimum over both ways to choose the classes A,B and all bipartite labelings $l()$ of the number of those edge labels in $1, \ldots, n-1$ that are not induced by $l()$.

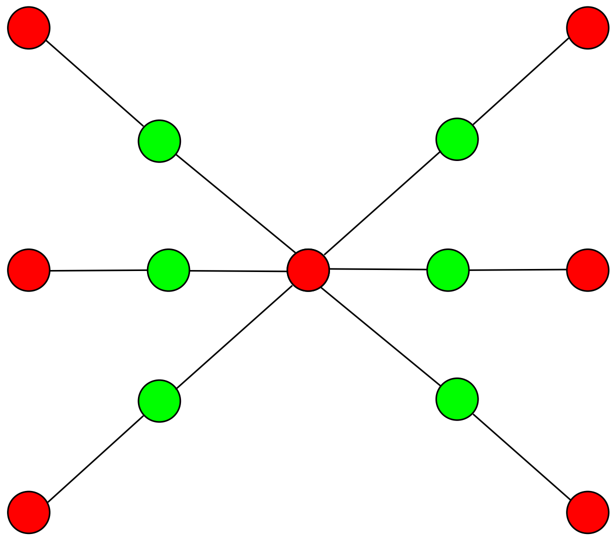
Two times label "5" but label "2" missing



Not very exciting: some trees have deficit 0, others positive deficit and even with maximum degree 3 soon deficitary trees occur. . .



Also trees with α-deficit larger than 1 occur:



But sometimes you need more points to make a good picture. . .

Maybe we need α-deficits for a lot of trees to see some structure. . .

The algorithm to compute the α-deficit

a standard branch and bound algorithm

- try to restrict the labeling in a way that doesn't change the deficit
- label in a way so that especially on low levels the recursion tree has few branches
- take care of symmetry

- try to restrict the labeling in a way that doesn't change the deficit

---

Fix an arbitrary of the two classes as A.

If the vertices in class A have original identifiers $a_1 < a_2 < \ldots < a_{|A|}$ one can restrict the search to labelings $l()$ with $l^{-1}(1) < l^{-1}(|A|)$.

Folklore: the deficit doesn't change under these restrictions

- label in a way that especially on low levels the backtrack tree has few branches

---

recursion over edge labels

$n-1$: 1 possibility: vertex labels $(1,|V|)$

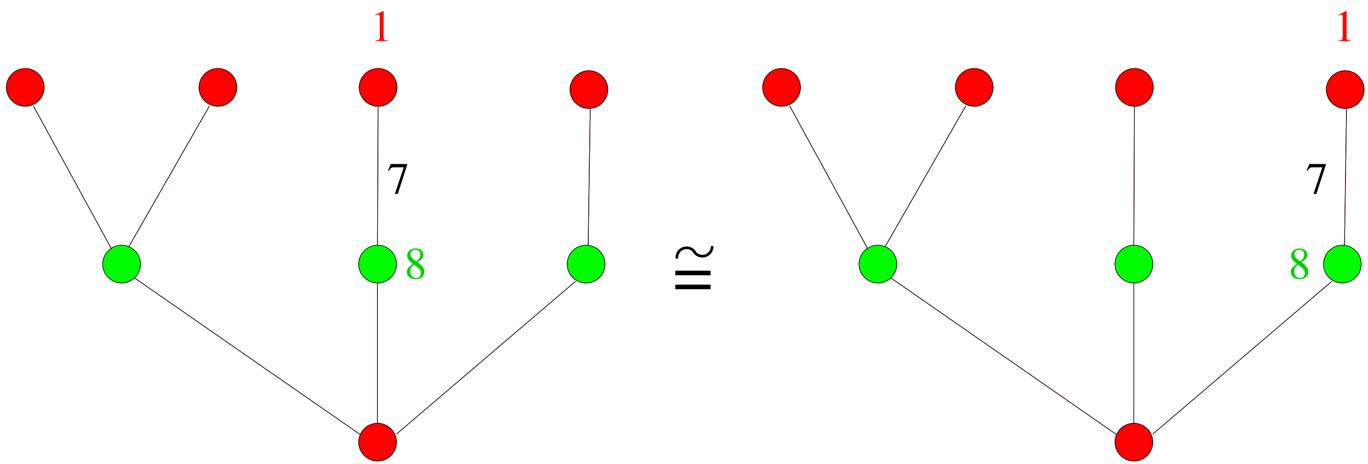
1: 1 possibility: $(|A|,|A|+1)$

$n-2$: 2 possibilities: $(1,|V|-1)$,$(2,|V|)$

2: 2 possibilities: $(|A|,|A|+2)$, $(|A|-1,|A|+1)$

...

- take care of symmetry (with nauty)



But: computing the symmetry for a huge number of partially labelled graphs is expensive. . .

- check whether trivial orbit was labelled
- switch to simpler computation as soon as only leafs are interchanged

We computed the α-deficit of more than 45.000.000.000 graphs. . .

. . . and these are the results:

Maximum degree 3

$\|V\|$	7	8	9	10	11	12	13	14	15	16	17
def. 1	1	1	1	2	0	1	0	0	1	0	0
def. 2	0	0	0	0	0	0	0	0	0	0	0

$\|V\|$	18	19	20	21	22	23	24	25	26	27
def. 1	0	0	0	0	0	2	0	0	0	0
def. 2	0	0	0	0	0	0	0	0	0	0

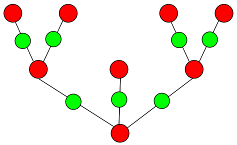
$\|V\|$	28	29	30	31	32	33	34	35	36
def. 1	0	0	0	6	0	0	0	0	0
def. 2	0	0	0	0	0	0	0	0	0

Maximum degree 3

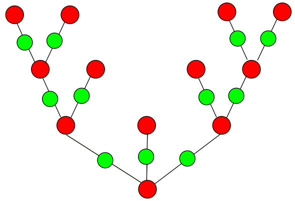
- The α-deficit seems to be at most 1.

- While for $|V| \leq 12$ no regularity in the vertex numbers allowing deficitary graphs can be seen, for $|V| > 12$ it seems as if deficitary graphs can only appear for $|V| = 7 + k * 8$ with $k \in \mathbb{N}$.

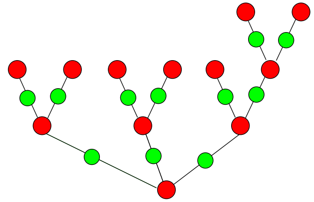
Let's look at the deficitary graphs:



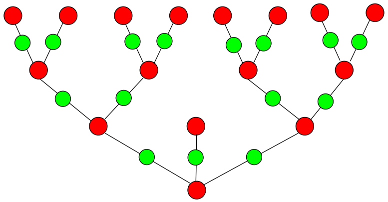
|V|=15



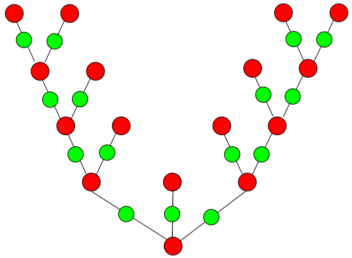
|V|=23



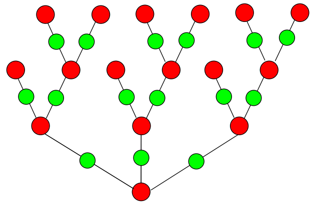
|V|=23



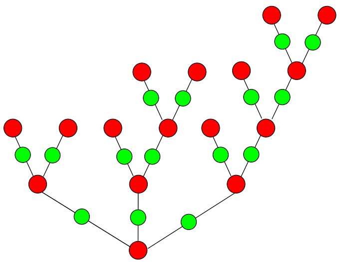
|V|=31



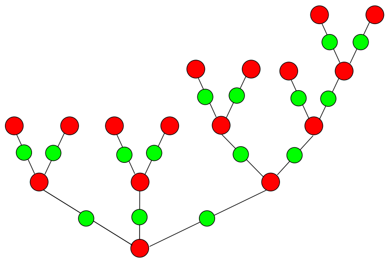
|V|=31



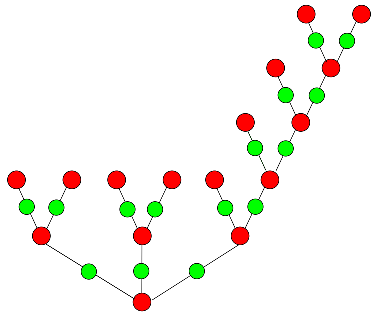
|V|=31



|V|=31



|V|=31



|V|=31

Maximum degree 4

$\lvert V\rvert$	8	9	10	11	12	13	14	15	16	17	18	19
def. 1	1	3	5	11	15	22	24	27	25	28	19	23

$\lvert V\rvert$	20	21	22	23	24	25	26	27	28	29	30
def. 1	6	15	2	15	2	26	0	63	0	152	0

Δ	3	4
$\alpha_{\text{def}}=1$	1	1

8 vertices

Δ	3	4	5
$\alpha_{\text{def}}=1$	1	3	1

9 vertices

Δ	3	4	5	6
$\alpha_{\text{def}}=1$	2	5	3	1

10 vertices

Δ	4	5	6	7
$\alpha_{\text{def}}=1$	11	7	3	1

11 vertices

Δ	3	4	5	6	7	8
$\alpha_{\text{def}}=1$	1	15	17	7	3	1

12 vertices

Δ	4	5	6	7	8	9
$\alpha_{\text{def}}=1$	22	30	18	7	3	1
$\alpha_{\text{def}}=2$			1			

13 vertices

Δ	4	5	6	7	8	9	10
$\alpha_{def} = 1$	24	58	34	18	7	3	1
$\alpha_{def} = 2$			1	1			

14 vertices

Δ	3	4	5	6	7	8	9	10	11
$\alpha_{def} = 1$	1	27	85	73	35	18	7	3	1
$\alpha_{def} = 2$				1	2	1			

15 vertices

Δ	4	5	6	7	8	9	10	11	12
$\alpha_{def} = 1$	25	133	124	78	35	18	7	3	1
$\alpha_{def} = 2$		1	1	3	2	1			

16 vertices

Δ	6	7	8	9	10	11	12	13
$\alpha_{\text{def}} = 1$	232	140	79	35	18	7	3	1
$\alpha_{\text{def}} = 2$	2	4	4	2	1			

17 vertices

Δ	7	8	9	10	11	12	13	14
$\alpha_{\text{def}} = 1$	275	142	80	35	18	7	3	1
$\alpha_{\text{def}} = 2$	7	6	4	2	1			

18 vertices

Δ	8	9	10	11	12	13	14	15
$\alpha_{\text{def}} = 1$	284	144	80	35	18	7	3	1
$\alpha_{\text{def}} = 2$	13	6	4	2	1			
$\alpha_{\text{def}} = 3$		1						

19 vertices

Δ	9	10	11	12	13	14	15	16
$\alpha_{def}=1$	291	144	80	35	18	7	3	1
$\alpha_{def}=2$	14	6	4	2	1			
$\alpha_{def}=3$	1	1						

20 vertices

Δ	10	11	12	13	14	15	16	17
$\alpha_{def}=1$	291	144	80	35	18	7	3	1
$\alpha_{def}=2$	14	6	4	2	1			
$\alpha_{def}=3$	2	1						

21 vertices

Δ	11	12	13	14	15	16	17	18
$\alpha_{def}=1$	292	144	80	35	18	7	3	1
$\alpha_{def}=2$	14	6	4	2	1			
$\alpha_{def}=3$	2	1						

22 vertices

Δ	10	11	12	13	14	15	16	17	18	19
$\alpha_{\text{def}} = 1$	1010	542	292	144	80	35	18	7	3	1
$\alpha_{\text{def}} = 2$	49	26	14	6	4	2	1			
$\alpha_{\text{def}} = 3$	3	3	2	1						

23 vertices

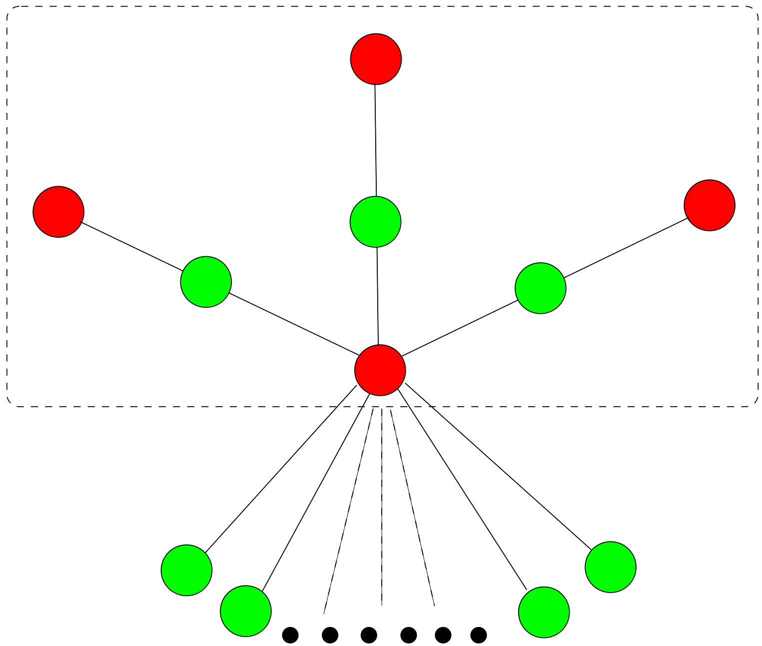
Δ	11	12	13	14	15	16	17	18	19	20
$\alpha_{\text{def}} = 1$	1020	544	293	144	80	35	18	7	3	1
$\alpha_{\text{def}} = 2$	49	26	14	6	4	2	1			
$\alpha_{\text{def}} = 3$	5	3	2	1						

24 vertices

Δ	12	13	14	15	16	17	18	19	20	21
$\alpha_{\text{def}} = 1$	1022	546	293	144	80	35	18	7	3	1
$\alpha_{\text{def}} = 2$	49	26	14	6	4	2	1			
$\alpha_{\text{def}} = 3$	5	3	2	1						

25 vertices

The unique deficitary tree with $\Delta = |V| - 4$:



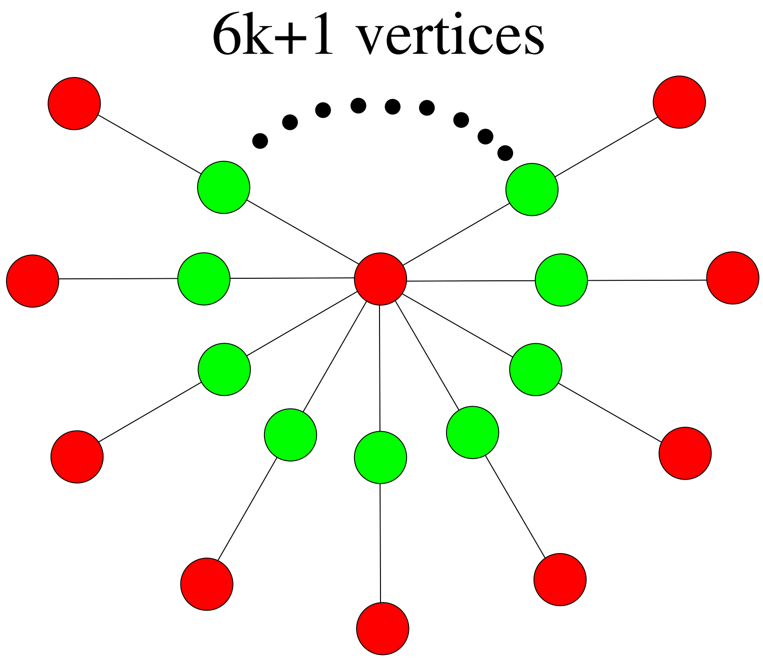
All the trees in the constant series are of the form

smallest graphs in the series plus a fan.

But adding a fan sometimes increases and sometimes decreases the deficit.

Why not here ?

Smallest graphs with $\alpha - deficit = k$



known: they have deficit k

So lots of things to prove. . .