The minimal perimeter for N deformable bubbles of equal area

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Elec. J. Combinatorics **17**:R45 (2010)

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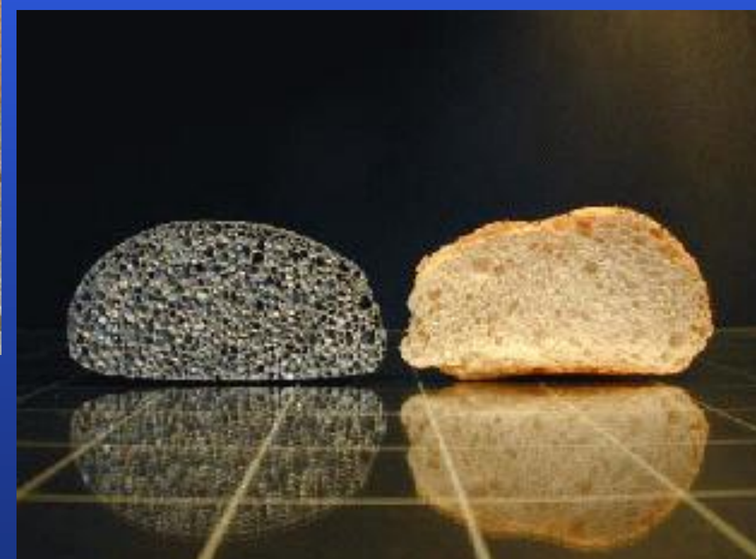
Why are foams interesting (to non-aphrologists)?

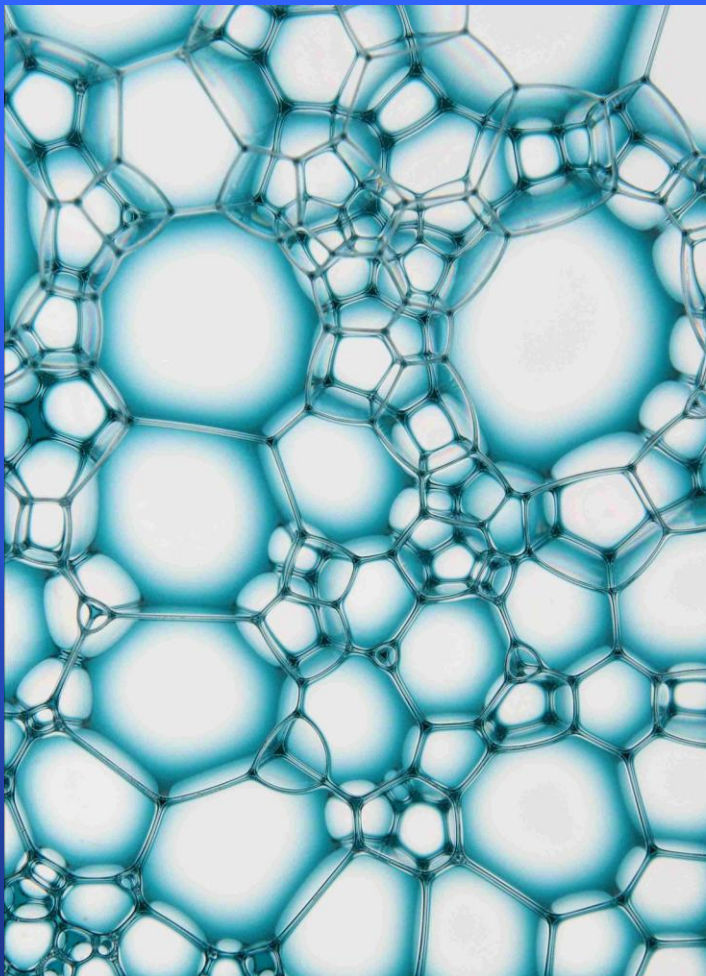
Many applications of industrial and domestic importance:

- Oil recovery
- Fire-fighting
- Ore separation
- (Industrial) cleaning
- Vehicle manufacture
- Food products



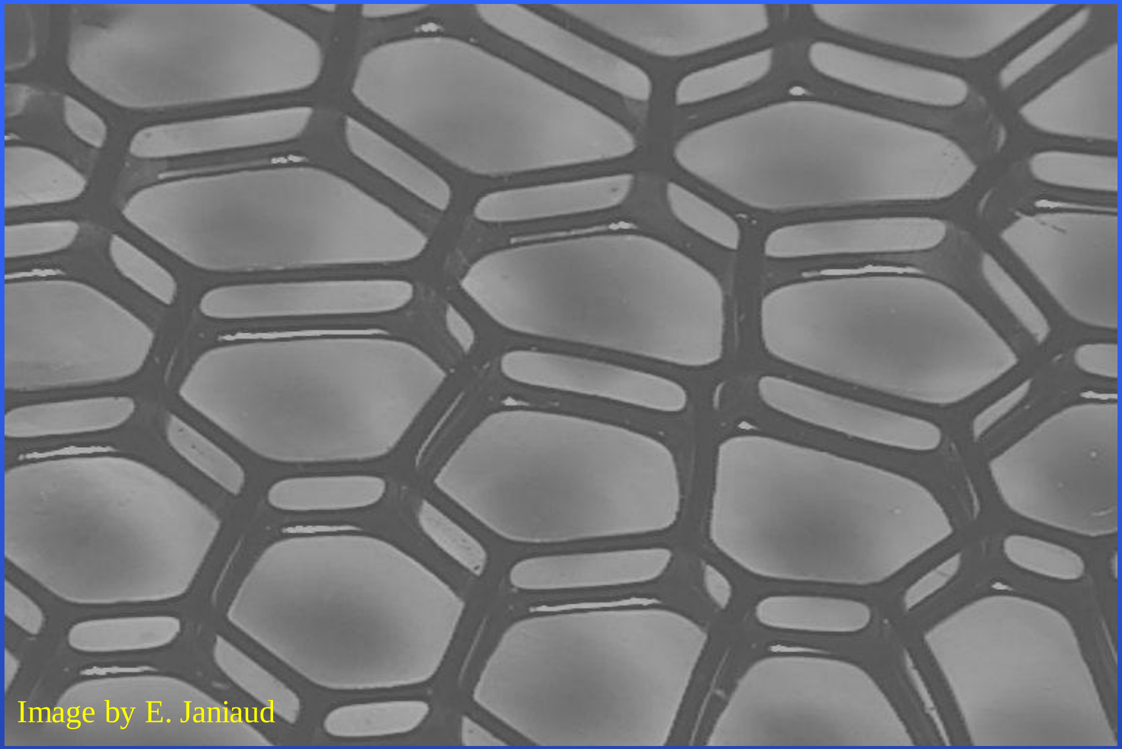
Highly concentrated emulsions are similar to foams.
Many solid foams are made from liquid precursors

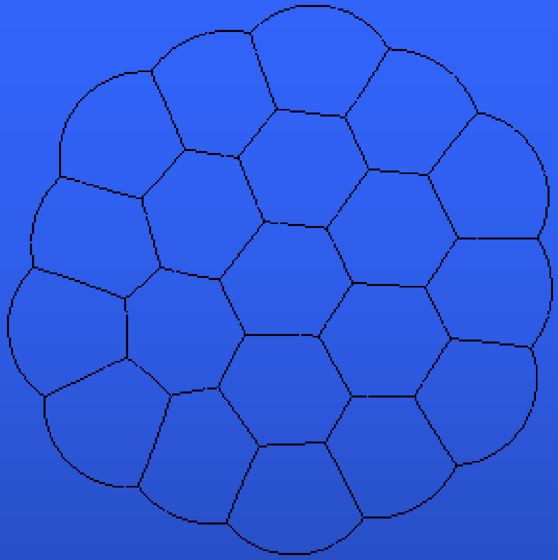
Foam Structure



- An equilibrium dry foam minimizes its surface area at constant volume.
- As a consequence (Plateau, Taylor) it is a complex fluid with special local geometry…
- Films meet **three-fold** at 120° angles in lines (Plateau borders), and the lines meet **tetrahedrally**.
- Laplace Law: film curvatures balanced by pressure differences, so each film has constant mean curvature.

Foam Structure in 2D (e.g. squeezed between glass plates)





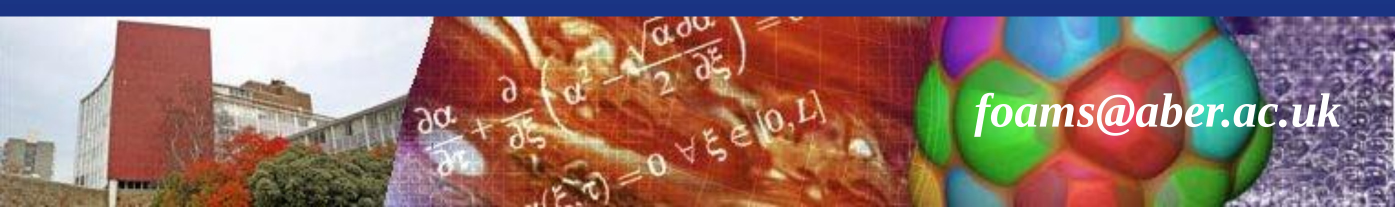
- A dry 2D foam at equilibrium minimizes perimeter and is 3-connected at 120º angles.
    - Each film is a circular arc.

Motivations for studying foam structure

- Mathematics: Each soap film is a minimal surface; provide solutions of isoperimetric problems.

- Physics: Dynamics of foams is largely dictated by the local static structure
  (e.g. stability, foamability, flow (rheology))

- Biology: “Bubbles” are a model for many cellular structures
  (e.g. drosophila eye, sea urchin skeleton, …)



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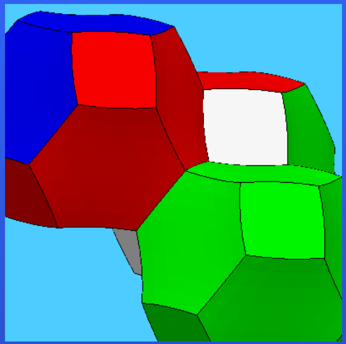
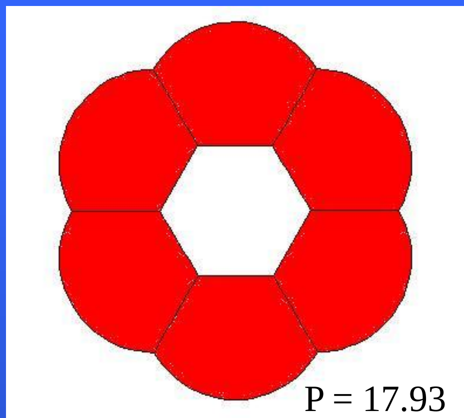
Least perimeter problems in foams

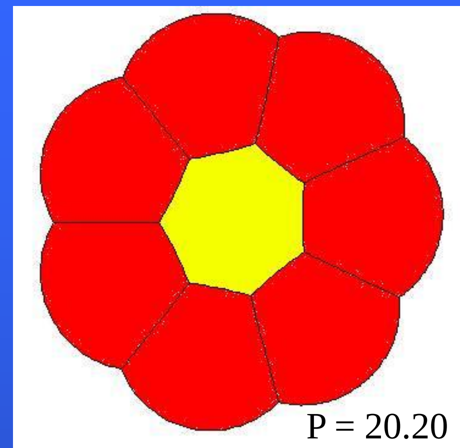
- What is the least perimeter division of the plane into equal area cells? Hexagonal honeycomb (Hales).

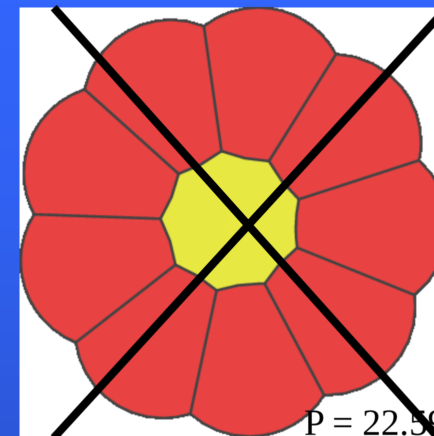


- 3D equivalent (Kelvin problem) unproven.
- Finite case: what is the arrangement of *N* cells of equal area/volume that minimizes the total perimeter/surface area?
- What effect does the shape of the boundary have?

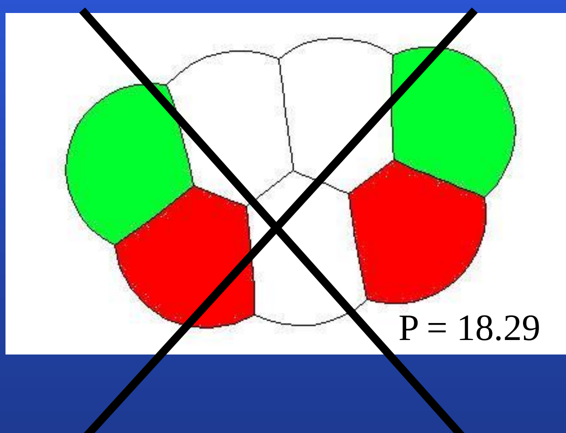
There are very many possibilities for each N, the perimeters vary only within a few percent, …

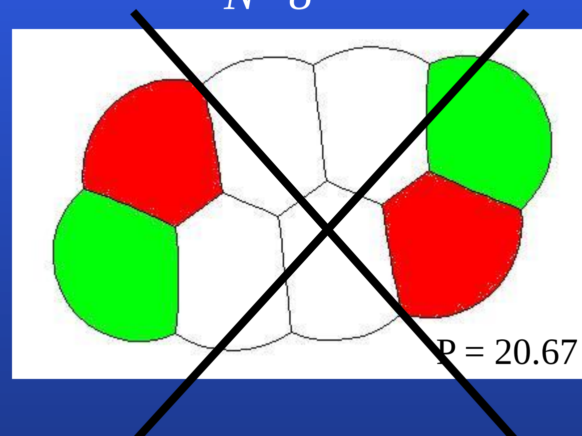


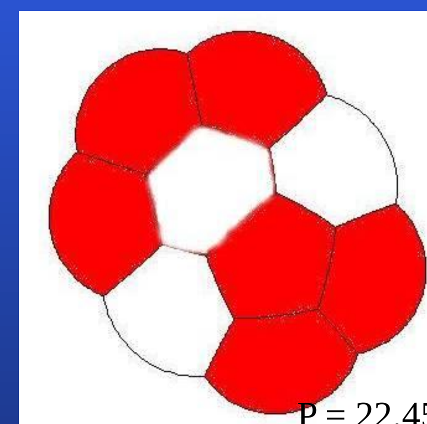




N=7 *N=8* *N=9*



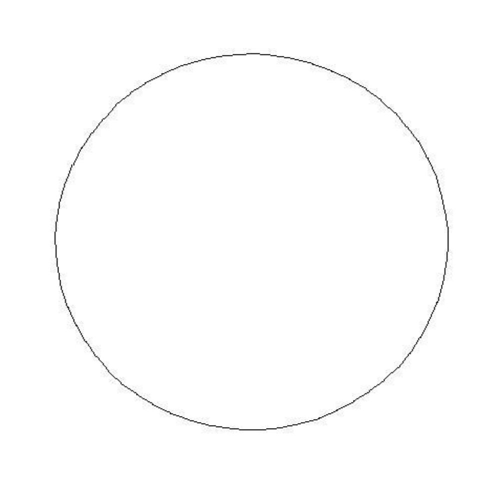




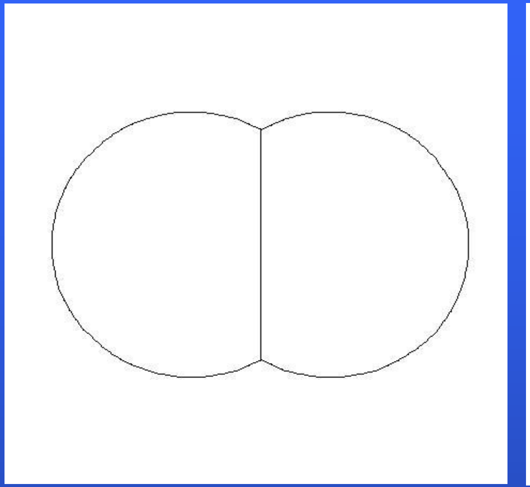
… and there appear to be few “patterns”.

Finite clusters with free boundary

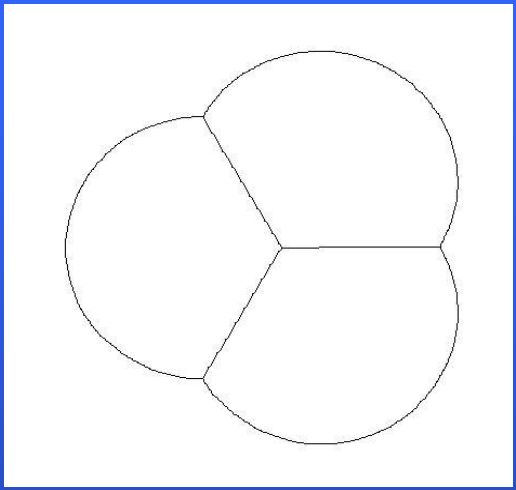
Proofs for N=1,2,3:



Isoperimetric problem



Morgan et al



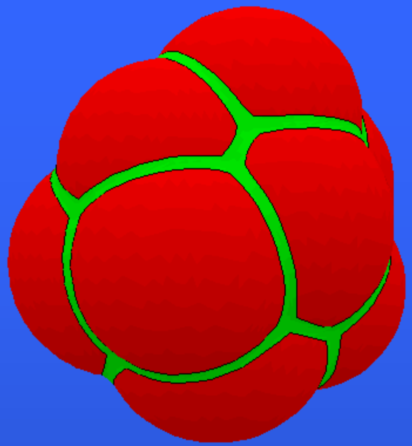
Wichiramala

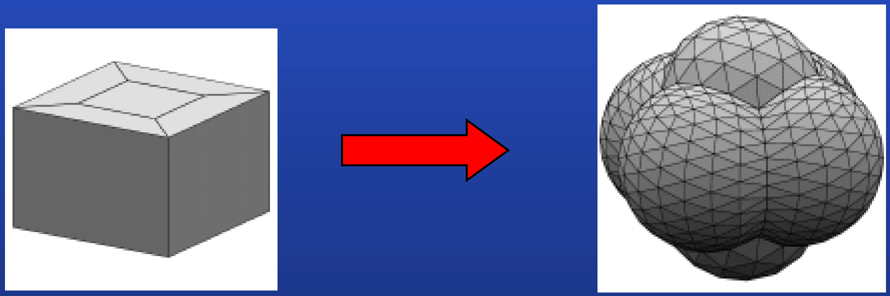
For larger *N*, instead of a proof, try many possibilities by "shuffling" clusters of N bubbles and choosing the best.

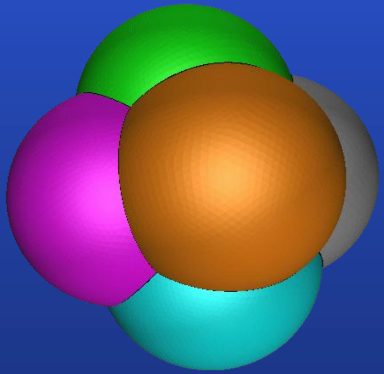
Cox et al. (2003) *Phil. Mag.* **83**:1393-1406

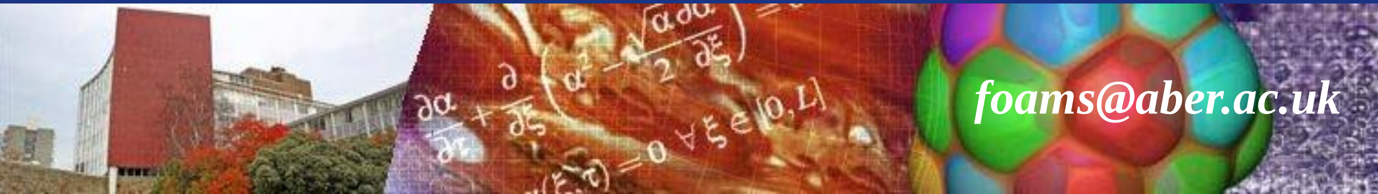
Simulating foam structure

Ken Brakke's Surface Evolver:
"The Surface Evolver is software expressly designed for the modeling of soap bubbles, foams, and other liquid surfaces shaped by minimizing energy subject to various constraints …"









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Colour scheme

- Colour bubbles according to number of sides (“charge”, q):

  bulk bubbles should be hexagonal: q=6-n;

  peripheral bubbles should be pentagonal: q=5-n.
- Total charge is 6 – how is it distributed?

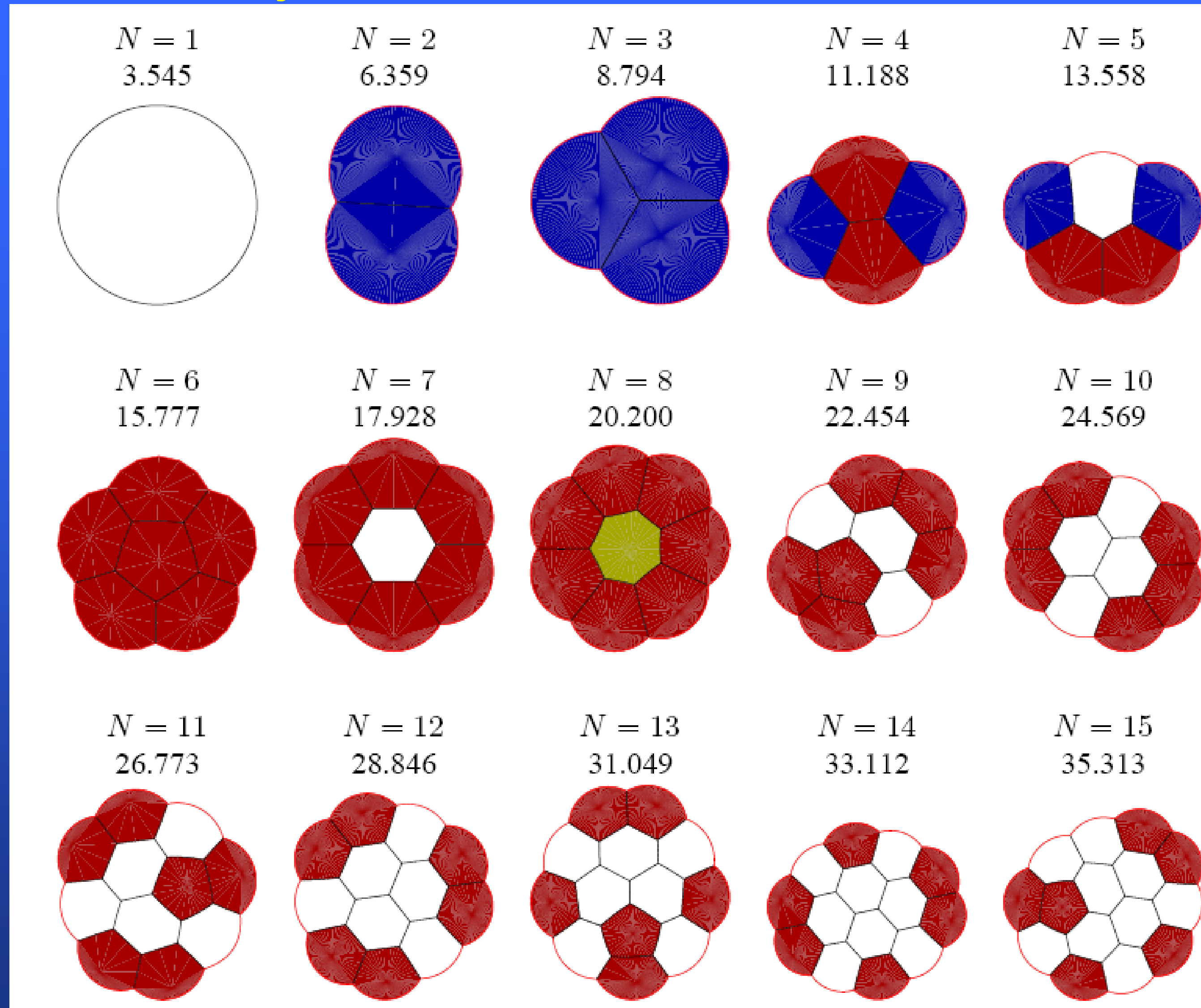
Cox et al. (2003) *Phil. Mag.* **83**:1393-1406

Finite clusters with free boundary

Never more than one negative (yellow) defect for N>5.

Positive defects mostly confined to the periphery.

Magic ``hexagonal'' numbers.



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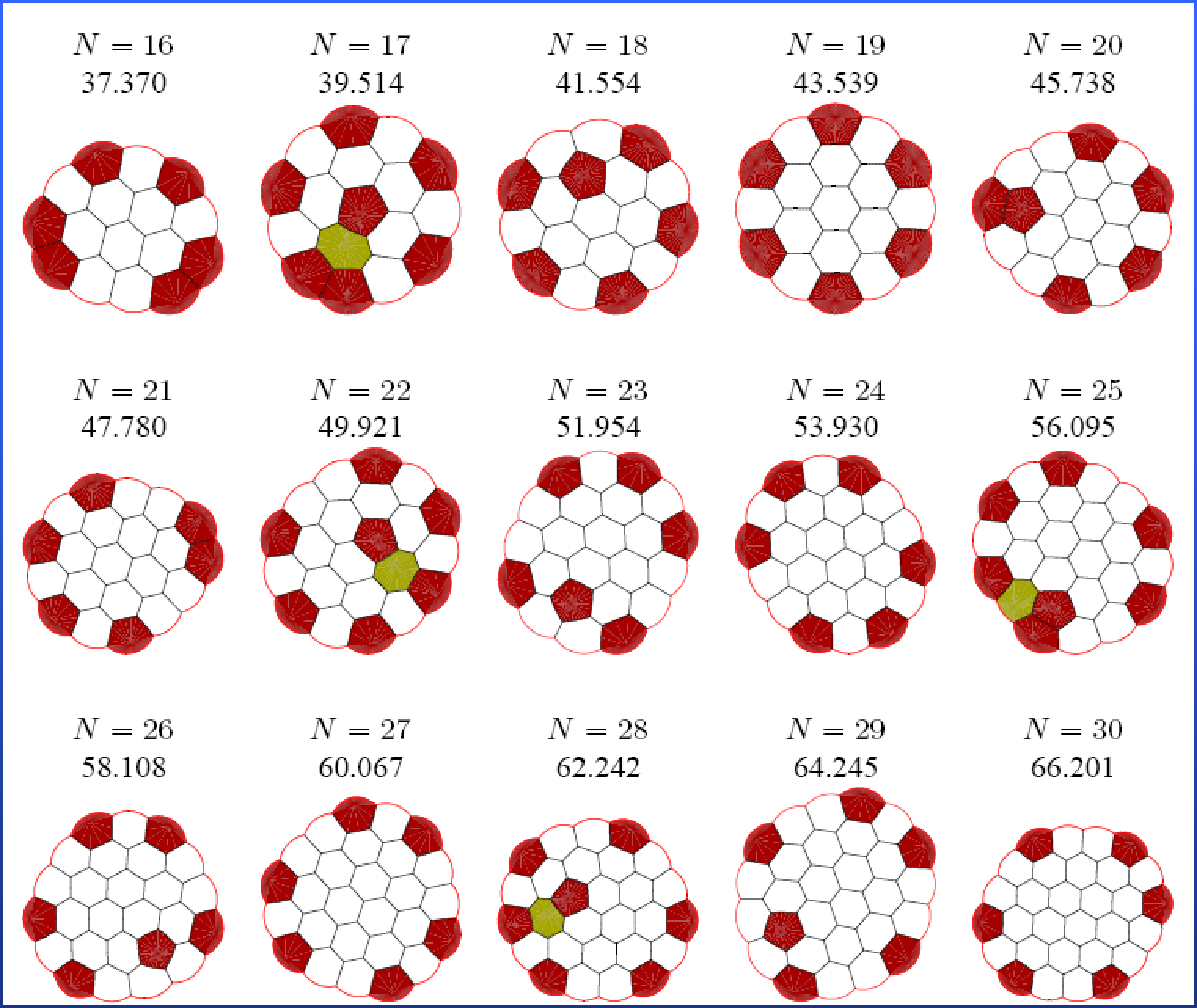

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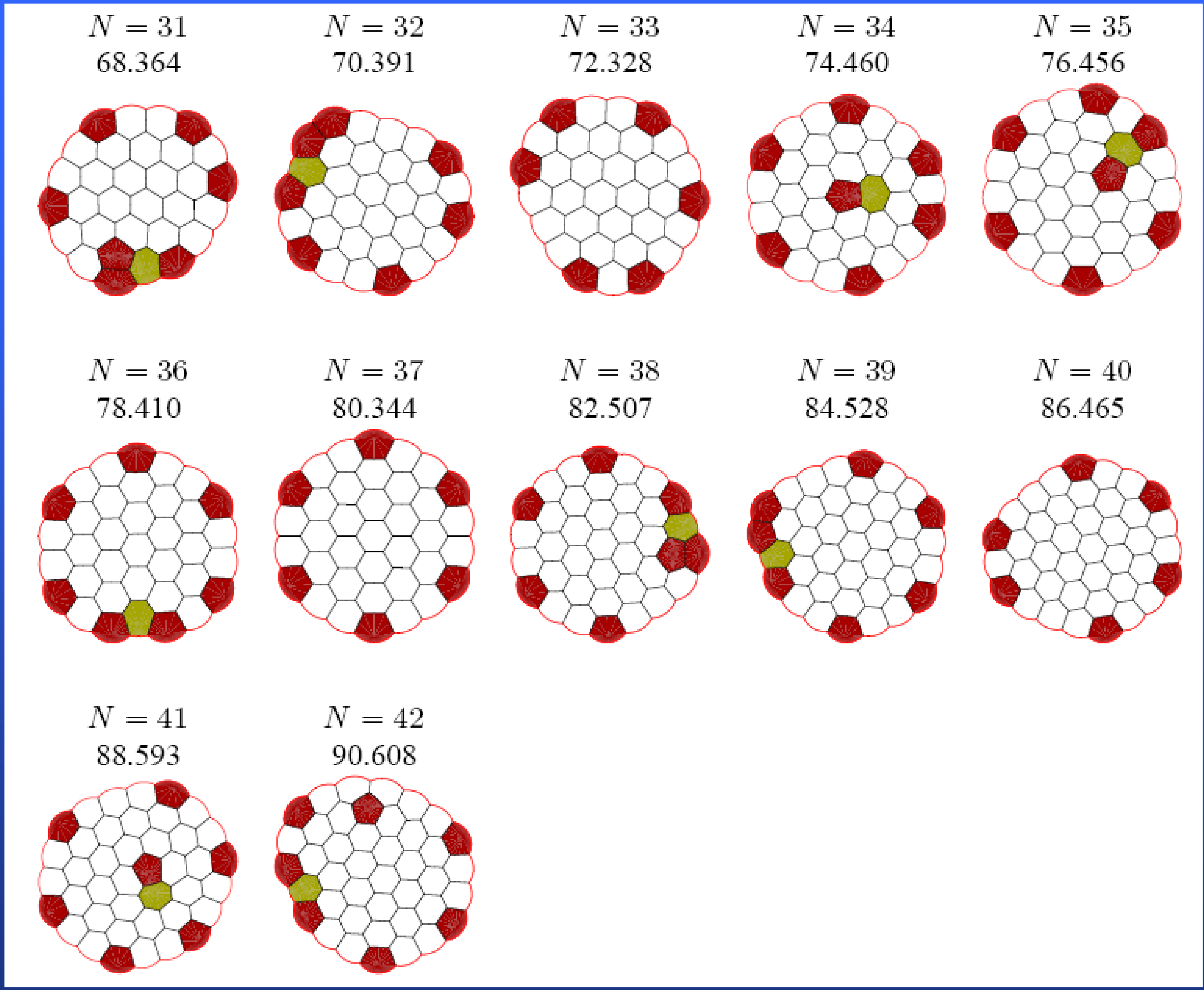
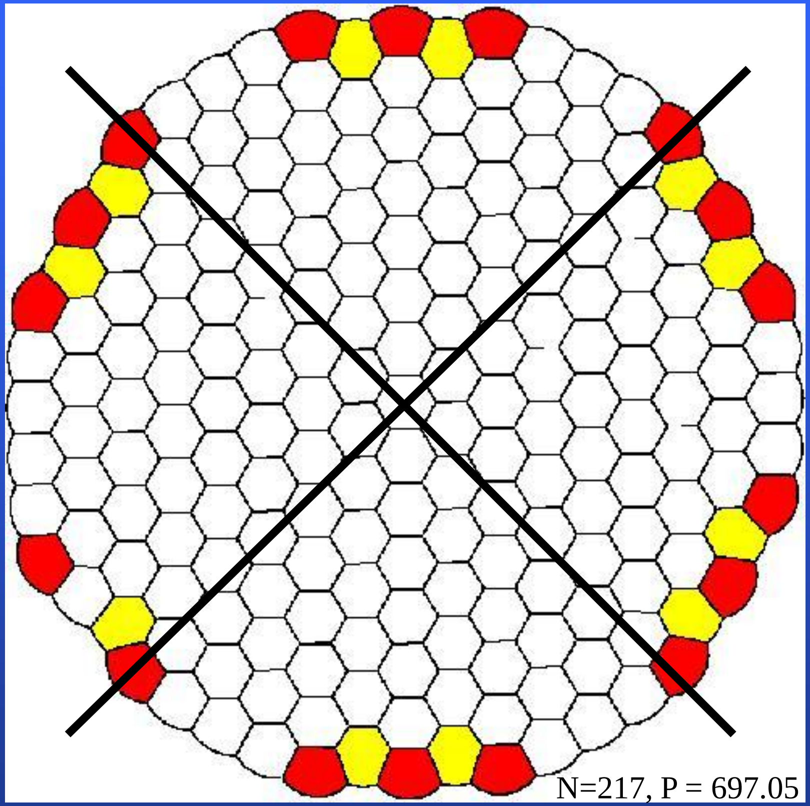
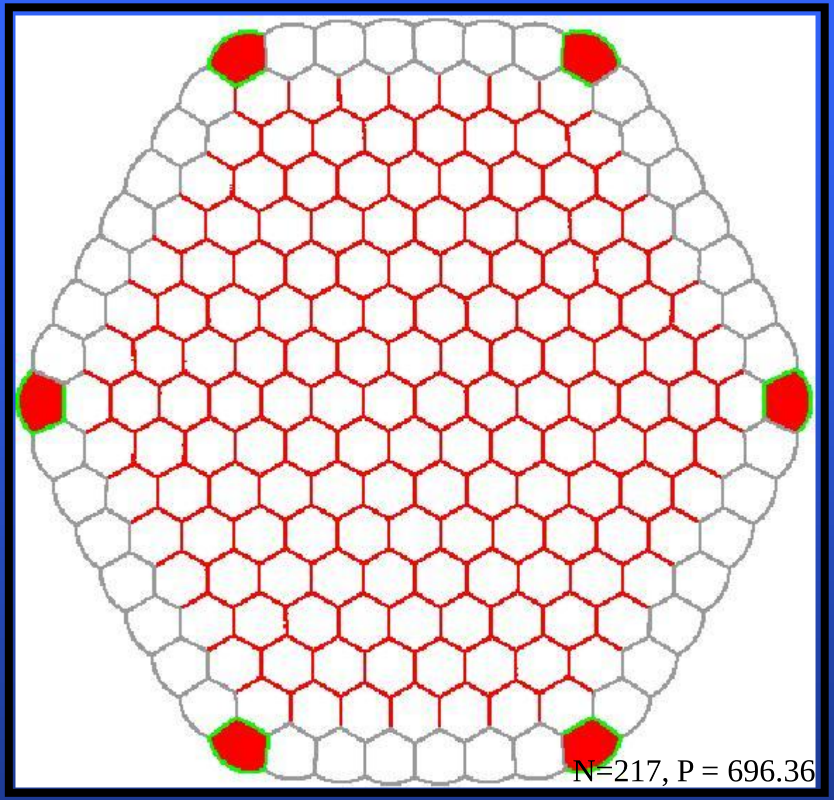


Effect of boundary shape at large *N*

Honeycomb structure in bulk: what shape should surface take?





Cox & Graner, Phil. Mag. (2003)

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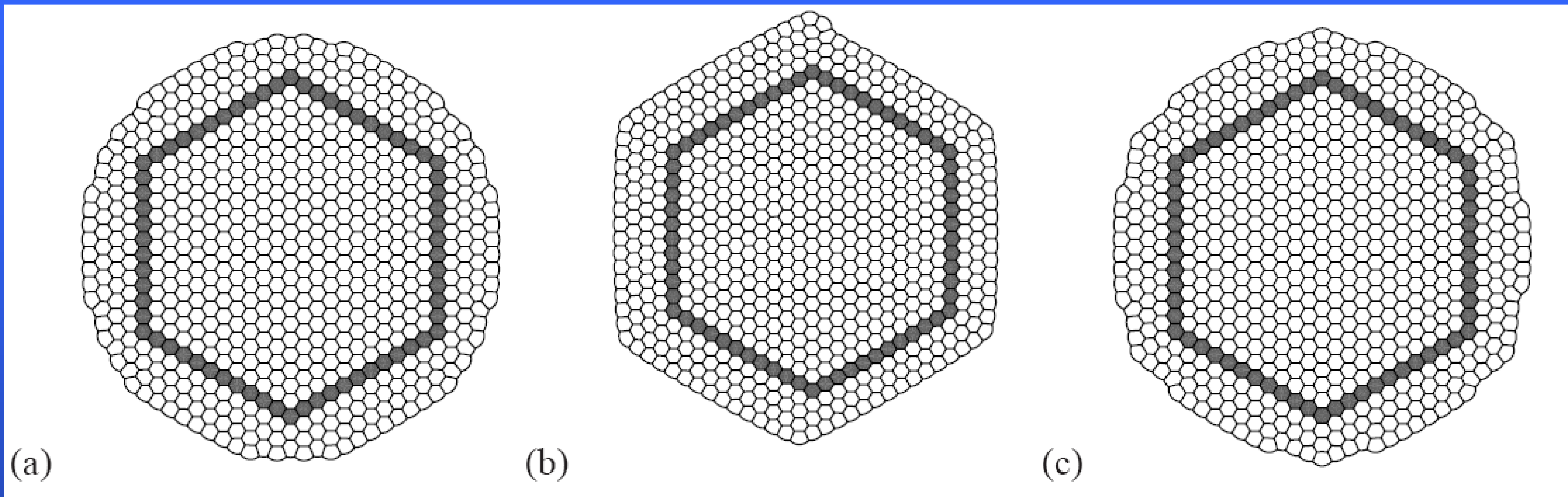
Effect of boundary shape at large *N*

Try three different arrangements for each *N*:

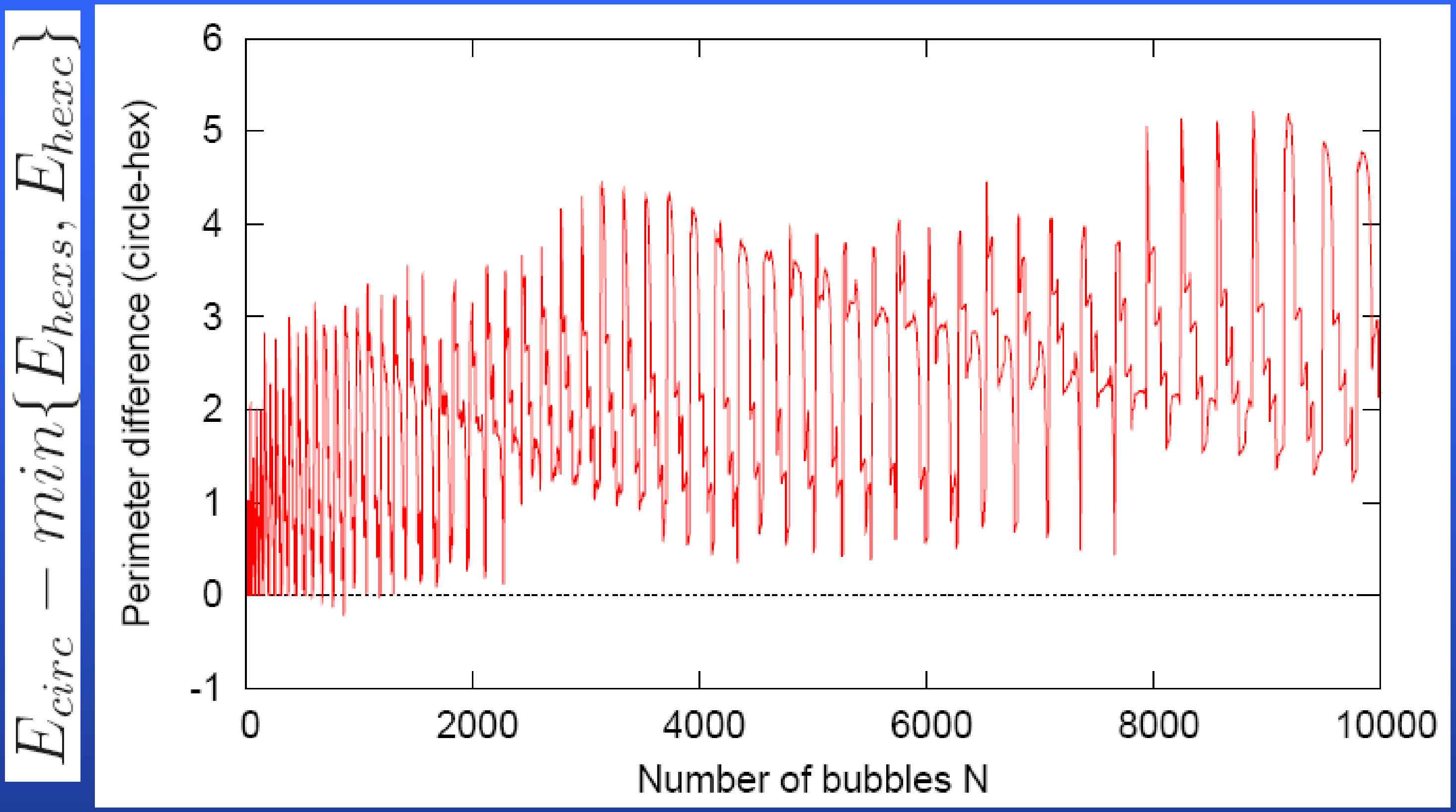


(a) Circular cluster: The bubble whose centre is farthest from the centre of the cluster is eliminated.

(b) **Spiral Hexagonal cluster:** the outer shell is eroded sequentially in an anticlockwise manner starting from the lowest corner

(c) **Corner hexagonal cluster**: the corners of the outer shell are first removed and the erosion proceeds from all of the six corners.

Effect of boundary shape at large N



A circular cluster appears to get worse as N increases

The circular cluster has lower perimeter in 20 out of 10,000 cases

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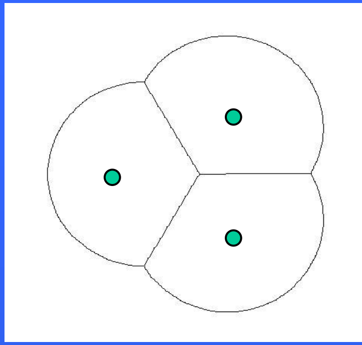
Potential correspondence?



Each bubble has a well-defined centre

(e.g. average of vertex positions)

Could there be a correspondence between the position of particles that minimize an inter-particle potential and the centres of the bubbles?

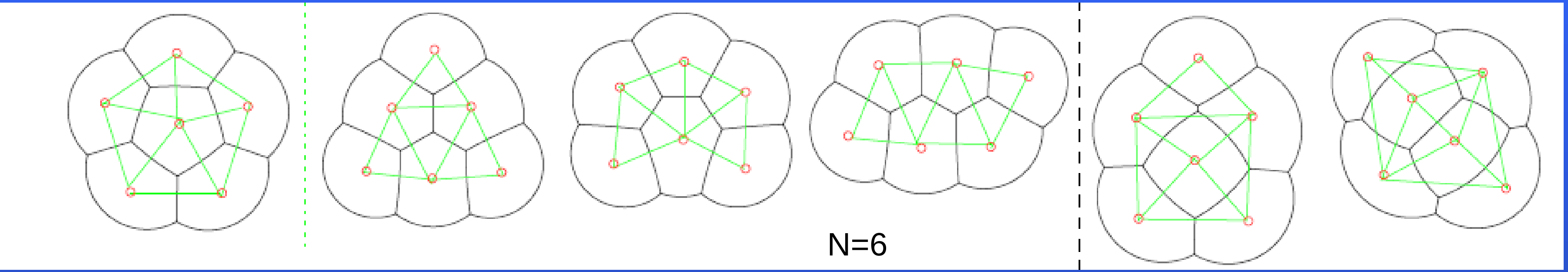
e.g. Quadratic confining potential, Coulomb potential, conjugate gradient and Voronoi construction, then Surface Evolver:

$$V = c_1 \sum_i \vec{r}_i^{\,2} + \sum_i \sum_{j \neq i} \frac{1}{|\vec{r}_i - \vec{r}_j|}$$

Different potentials find optimal candidates for different N, some better than the undirected "shuffling", but no single potential finds all.

Towards a proof … graph enumeration?

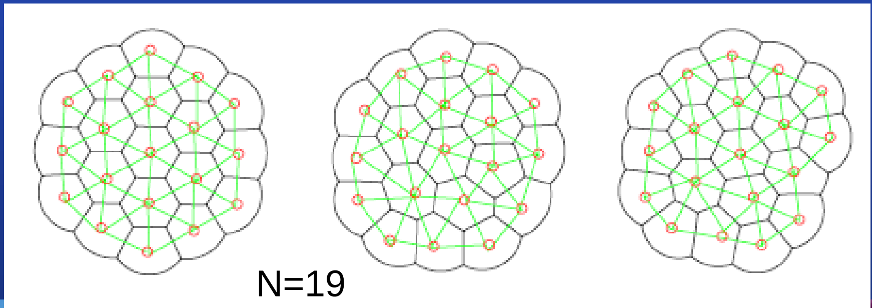
Each edge of the cluster defines a link between centres … so construct the dual graph:



Could we enumerate all possible convex planar graphs with *N* vertices, with conditions on the degree of internal and peripheral vertices?

$$5 \le n^i \le 7 \quad \text{and} \quad 3 \le n^p \le 5$$

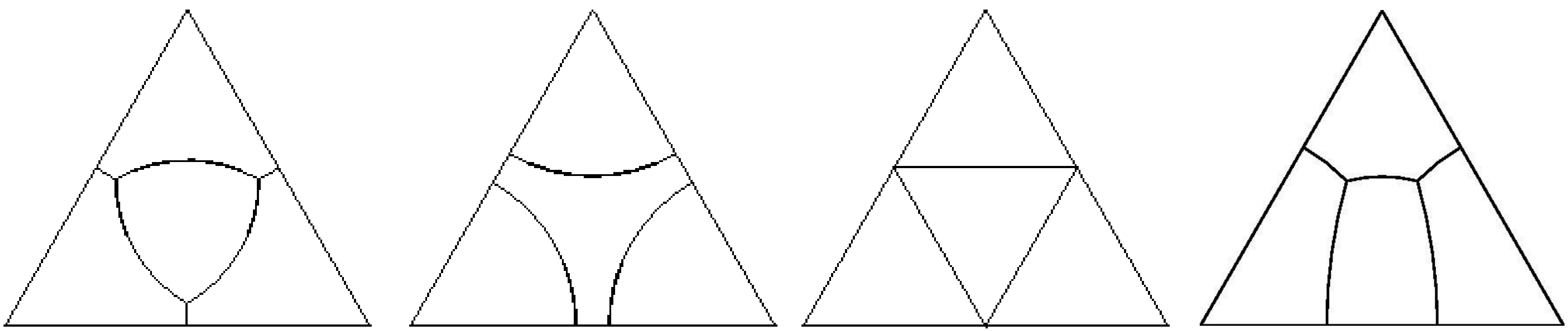
e.g. plantri/cage?



Confined clusters

Confine the foam within a **fixed boundary** and search for the least perimeter arrangement of bubbles.

e.g. equilateral triangle:



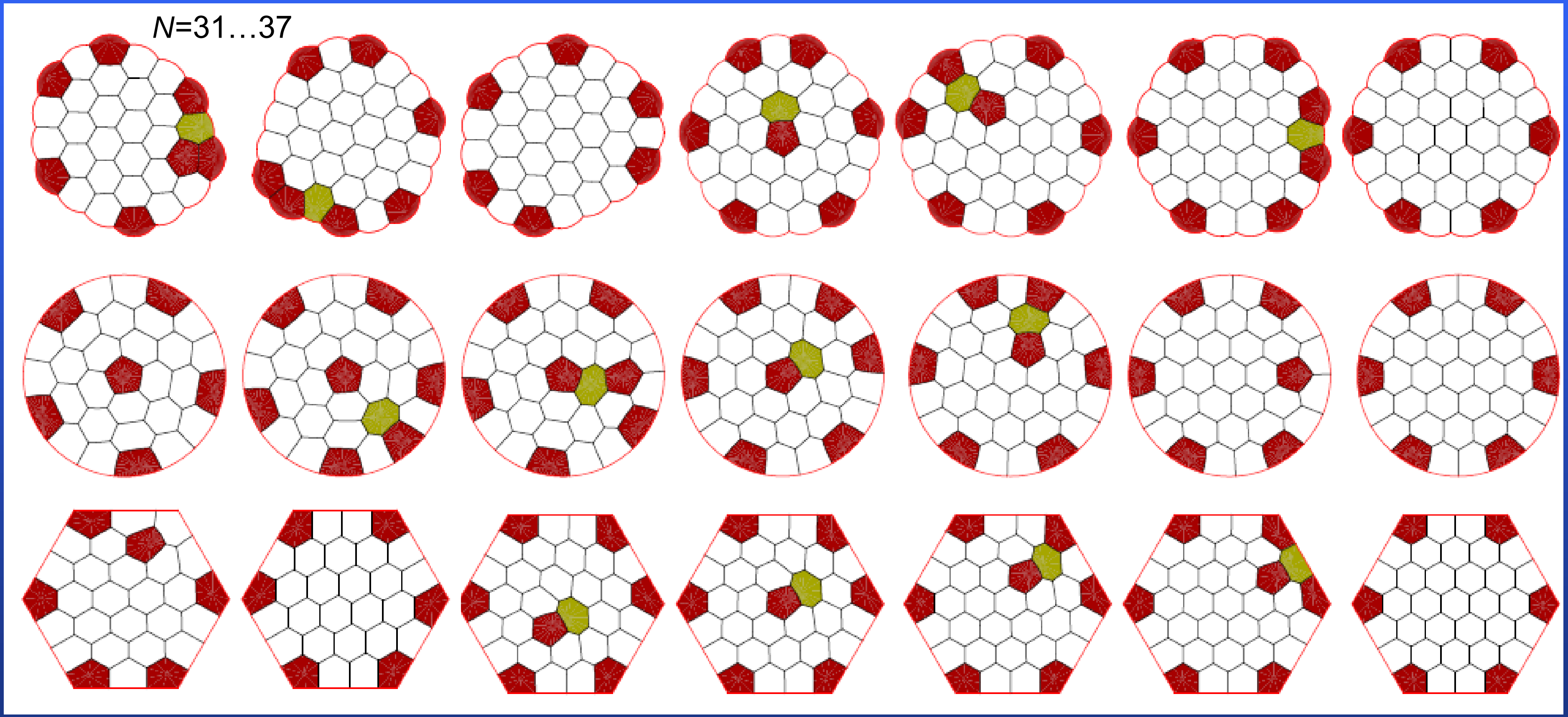
P = 4.4185   P = 4.4285   P = 4.5000   **P = 4.305**

Ben Shuttleworth, MMath 2008
proof by enumeration of connected candidates

Intuition not always the best guide: use potential search procedure …

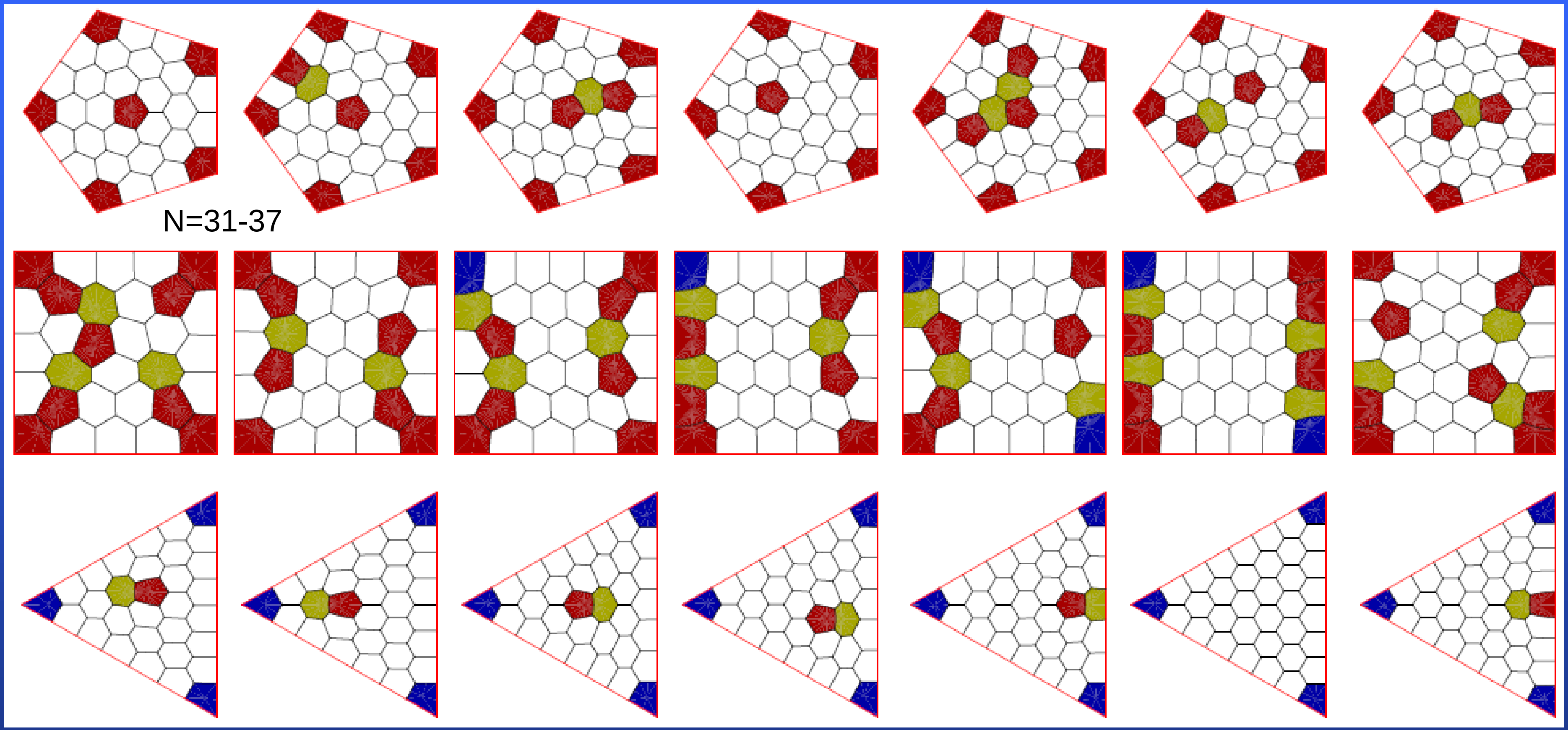
Confined clusters

Having found an optimal candidate for the free case, for which **fixed boundary shapes** does it remain optimal?



Confined clusters

Change confining potential to create different initial conditions



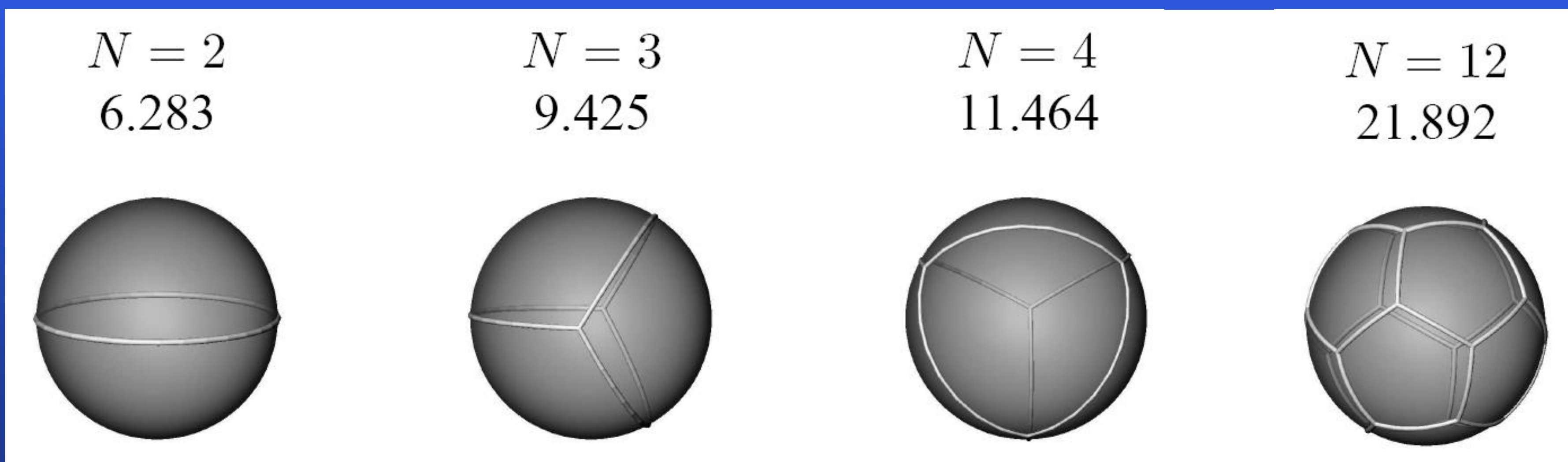
Note the pattern for a triangular boundary – almost replicated for a hexagonal boundary

Clusters confined to the surface of a unit sphere

Which configuration of equal area cells realizes the least perimeter?

Retain 120º angles, but edges not arcs for *N*=11, *N* >12.

Proofs for *N* up to 4, and *N*=12.

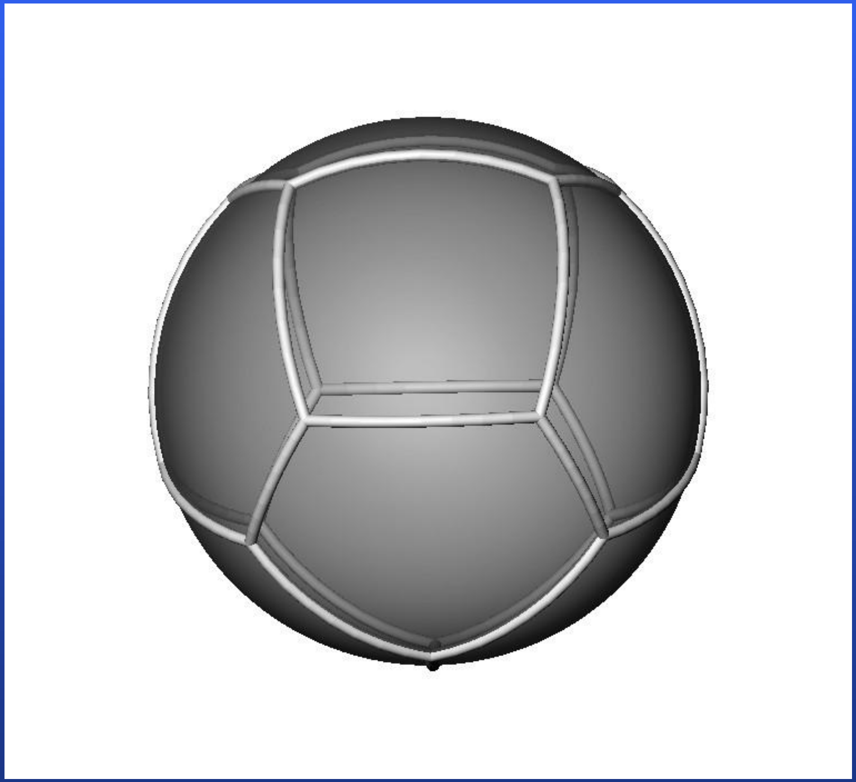


Clusters confined to the surface of a unit sphere

Random shuffling procedure gives good results for $N<20$.

For example:

$N=11$ is lowest to have a hex face



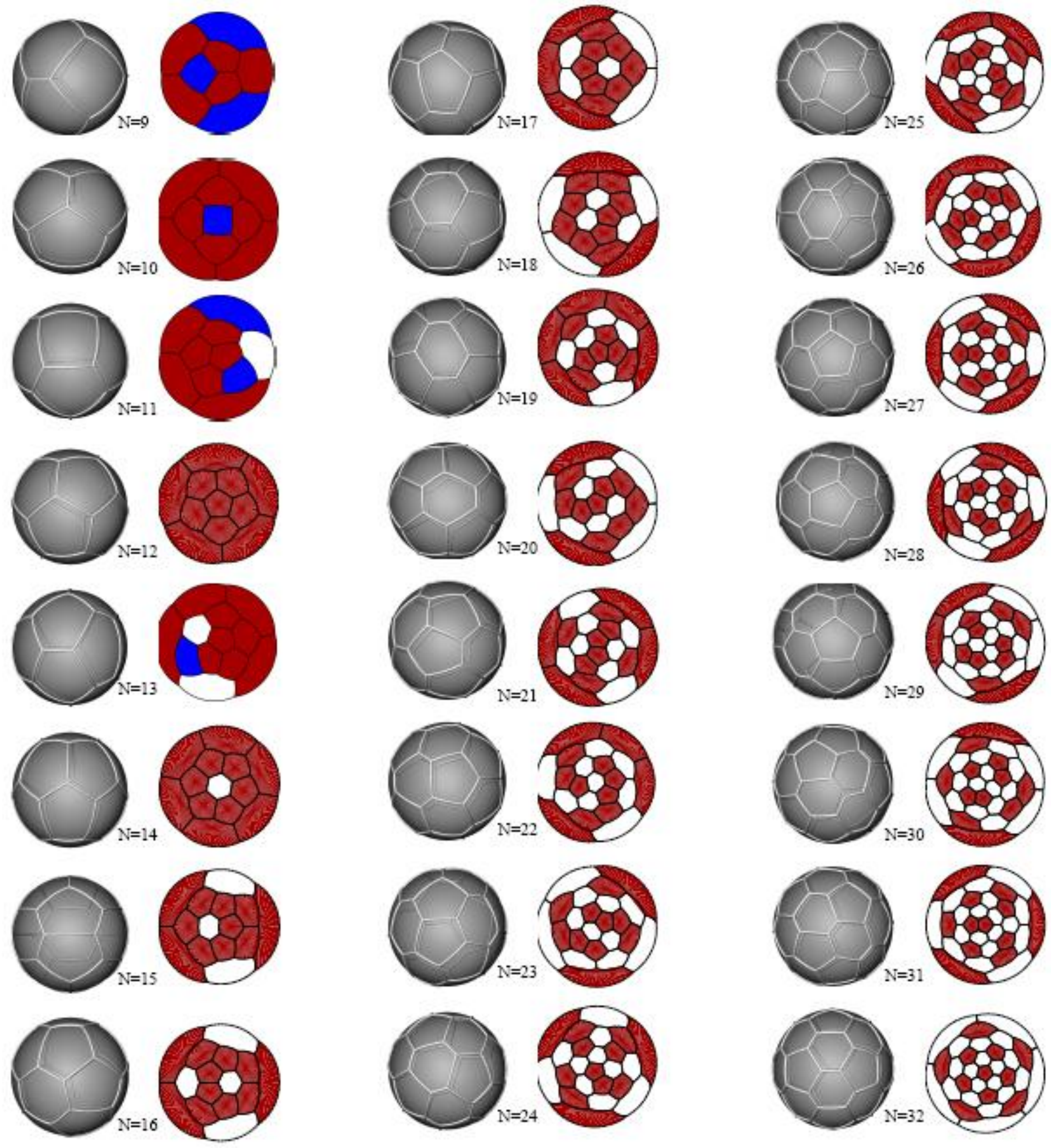
$N=13$ is highest to have a quad face

Clusters confined to the surface of a unit sphere

For $14 \leq N \leq 20$ find that optimal candidate consists only of pentagons and hexagons.

cf. fullerenes

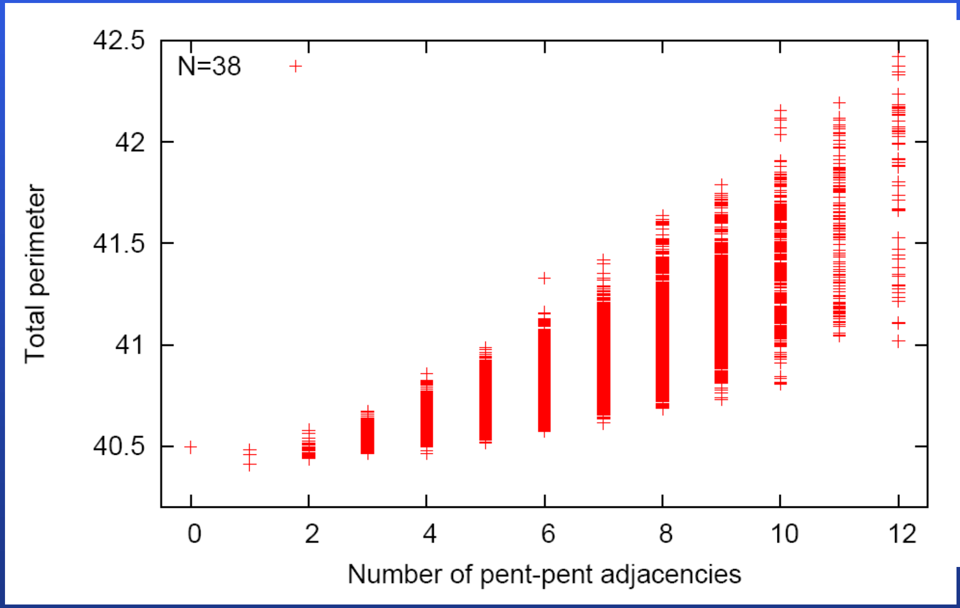
For $N \geq 20$ enumerate all tilings with 12 pentagons and N-12 hexagons using *Cage*.

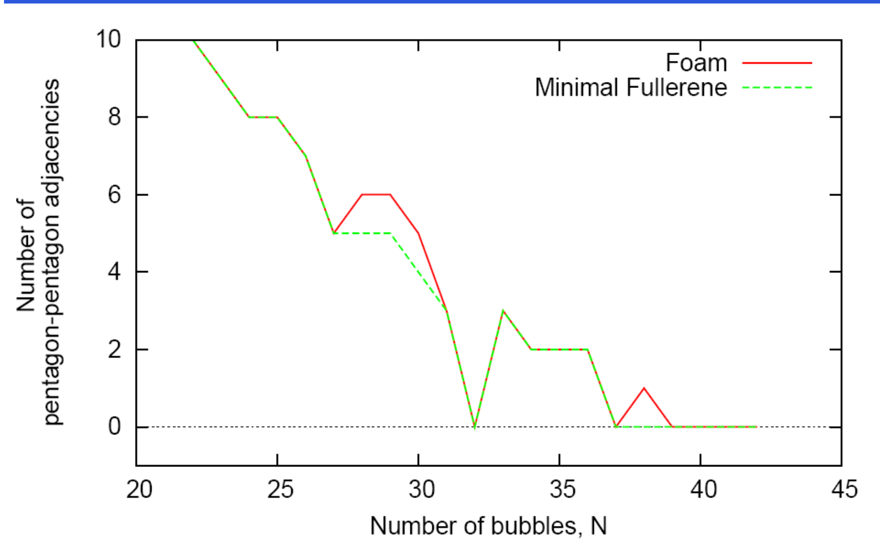


Clusters confined to the surface of a unit sphere

Conjecture that for $N > 13$ need to find the most widely-spaced arrangement of pentagons

Cox & Flikkema, Elec. J. Combinatorics 17:R45 (2010)





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Open questions

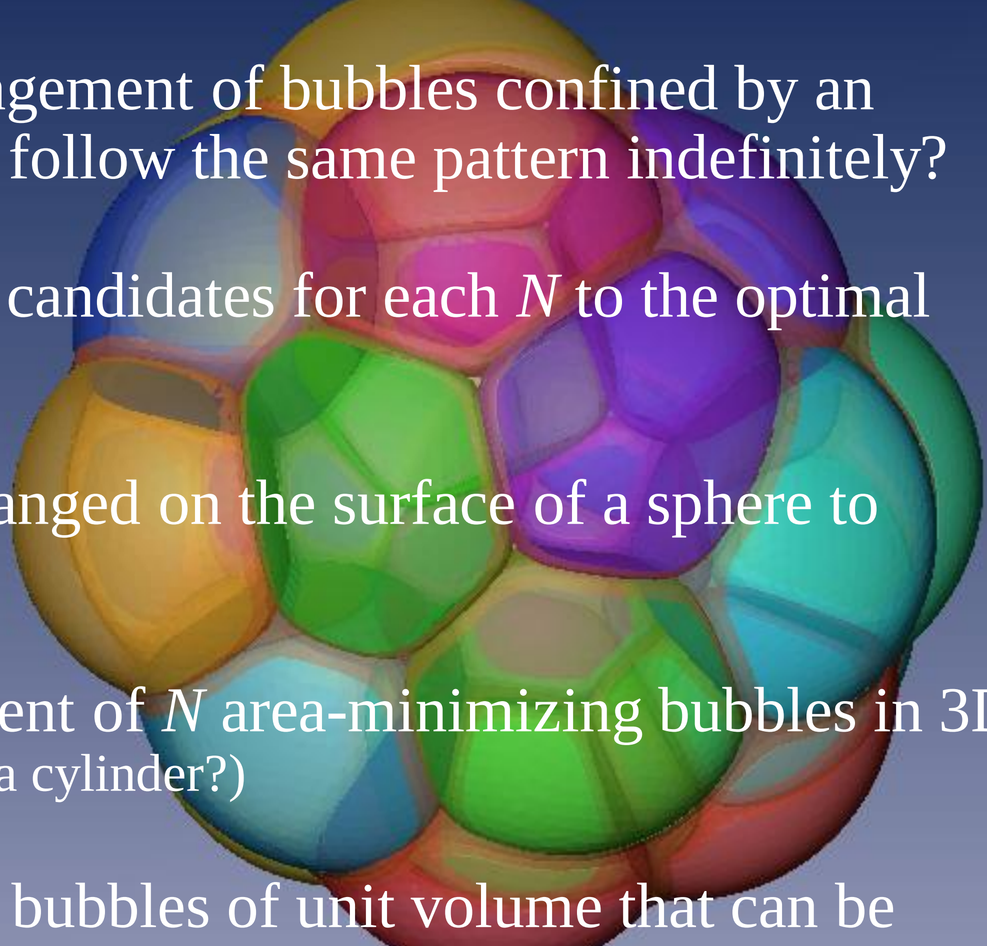
- Does the least perimeter arrangement of bubbles confined by an equilateral triangular boundary follow the same pattern indefinitely?

- Is it possible to enumerate all candidates for each *N* to the optimal free/confined cluster in 2D?

- How should pentagons be arranged on the surface of a sphere to minimize perimeter?

- What is the optimal arrangement of *N* area-minimizing bubbles in 3D? (Free? Confined within a sphere? Or a cylinder?)

- What is the largest number of bubbles of unit volume that can be packed around one other? (Kissing conjecture: 12 in 2D, 32 in 3D.)



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Kissing problem for bubbles