8 families of $(\{a, b\}, k)$-spheres: fullerenes $(\{5, 6\}, 3)$- and 7 analogs

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2. Polyhedrality of $(\{a, b\}, k)$-spheres
3. 8 families: four smallest members
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6. Parameterizing $(\{a, b\}, k)$-spheres
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I. 8 infinite families of $(\{a, b\}, k)$-spheres

Non-hyperbolic (R, k)-spheres

- Given $R \subset \mathbb{N}$, an (R, k)-sphere S is a k-regular map on the sphere whose faces have *gonalities* (numbers of sides) $i \in R$.
- Let v, e and $f = \sum_i p_i$ be the numbers of vertices, edges and faces of S, where p_i is the number of i-gonal faces.
  Clearly, k-regularity implies $kv = 2e = \sum_i ip_i$ and
- Euler formula $2 = v - e + f = \frac{2e}{k} - e + f = \frac{2-k}{k}e + \sum_i p_i = \sum_i p_i \left(\frac{i(2-k)}{2k} + 1\right)$ become $4k = \sum_i p_i(2k - i(k-2))$.

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- Let us see $2k - i(k-2)$ as the curvature of i-gonal faces and Euler formula as equality of the total curvature to $4k$.
- We consider only non-hyperbolic maps, i.e. $\frac{1}{k} + \frac{1}{m} \geq \frac{1}{2}$ for $m = \max\{i \in R\}$. So, $m \leq \frac{2k}{k-2}$ and the family of (R, k)-maps can be infinite only for $m = \frac{2k}{k-2}$ when p_m is not restricted.
- Then, clearly, all possible (m, k) are $(6, 3)$, $(4, 4)$, $(3, 6)$.

8 families of $(\{a,b\},k)$-spheres

- An $(\{a,b\},k)$-sphere is an (R,k)-sphere with $R=\{a,b\}$, $1 \le a < b$. It has $v = \frac{1}{k}(ap_a + bp_b)$ vertices.
- We have $b = \frac{2k}{k-2}$; so, $(b,k) = (6,3), (4,4), (3,6)$ and Euler formula become

$$\begin{array}{rcl} 12 = \sum_i (6-i)p_i & \text{if} & k=3 \\ 8 = \sum_i (4-i)p_i & \text{if} & k=4 \\ 6 = \sum_i (3-i)p_i & \text{if} & k=6 \end{array}$$

- Further, $p_a = \frac{2b}{b-a}$ and all possible (a, p_a) are: $(5,12), (4,6), (3,4), (2,3)$ for $(b,k)=(6,3)$; $(3,8), (2,4)$ for $(b,k)=(4,4)$; $(2,6), (1,3)$ for $(b,k)=(3,6)$.

8 families of $(\{a, b\}, k)$-spheres

- Those 8 families can be seen as spheric analogs of the regular plane partitions $\{6^3\}$, $\{4^4\}$, $\{3^6\}$ with p_a a-gonal "defects", *disclinations* added to get the curvature of the sphere $\mathbb{S}^2$.

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- $(\{5, 6\}, 3)$-spheres are (geometric) fullerenes, of great practical interest. $\{5, 6\}_{60}$ is a new form C_{60} of a carbon allotrope.
- $(\{a, b\}, 4)$-spheres are minimal projections of alternating links, whose components are their *central circuits* (those going only ahead) and crossings are the verices.

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- Let us denote $(\{a, b\}, k)$-sphere with v vertices by $\{a, b\}_v$.
- By smallest member Dodecahedron $\{5, 6\}_{20}$, Cube $\{4, 6\}_8$, Tetrahedron $\{3, 6\}_4$, Octahedron $\{3, 4\}_6$ and $3\times K_2$ $\{2, 6\}_2$, $4\times K_2$ $\{2, 4\}_2$, $6\times K_2$ $\{2, 3\}_2$, Trifolium $\{1, 3\}_1$, we call eight families: dodecahedrites, cubites, tetrahedrites, octahedrites and 3-bundelites, 4-bundelites, 6-bundelites, trifoliumites.

8 families: existence criterions

Grűnbaum-Motzkin, 1963: criterion for $k{=}3 \leq a$; Grűnbaum, 1967: for $(\{3,4\},4)$-spheres; Grűnbaum-Zaks, 1974: for other cases.

k	(a,b)	smallest one	it exists if and only if	p_a	v
3	$(5,6)$	Dodecahedron	$p_6 \neq 1$	12	$20+2p_6$
3	$(4,6)$	Cube	$p_6 \neq 1$	6	$8+2p_6$
4	$(3,4)$	Octahedron	$p_4 \neq 1$	8	$6+p_4$
6	$(2,3)$	$6 \times K_2$	p_3 is even	6	$2+\frac{p_3}{2}$
3	$(3,6)$	Tetrahedron	p_6 is even	4	$4+2p_6$
4	$(2,4)$	$4 \times K_2$	p_4 is even	4	$2+p_4$
3	$(2,6)$	$3 \times K_2$	$p_6{=}(k^2+kl+l^2)-1$	3	$2+2p_6$
6	$(1,3)$	Trifolium	$p_3{=}2(k^2+kl+l^2)-1$	3	$\frac{1+p_3}{2}$

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3	$(2,6)$	$3 \times K_2$	$p_6=(k^2+kl+l^2)-1$	3	$2+2p_6$
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$(\{3,6\},3)$- (Grűnbaum-Motzkin, 1963) and $(\{2,4\},4)$-spheres (Deza-Shtogrin, 2003) admit a simple 2-parametric description.

Generation of $(\{a, b\}, k)$-spheres

- $(\{2,3\},6)$-spheres, except $2 \times K_2$ and $2 \times K_3$, are the duals of $(\{3,4,5,6\},3)$-spheres with six new vertices put on edge(s). Exp: $(\{5,6\},3)$-spheres with 5-gons organized in six pairs.
- $(\{1,3\},6)$-spheres, except $\{1,3\}_1$ and $\{1,3\}_3$, are as above but with 3 edges changed into 2-gons enclosing one 1-gon.

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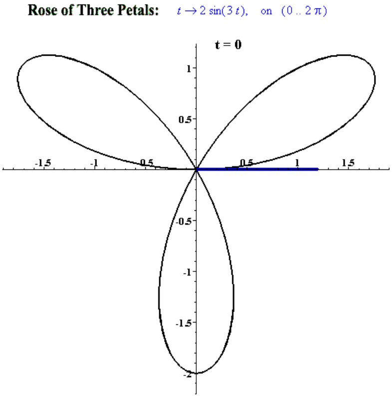
- $(\{2,6\},3)$-spheres are given by the *Goldberg-Coxeter construction* from $\text{Bundle}_3 = 3 \times K_2$ $\{2,6\}_2$.
- $(\{1,3\},6)$-spheres come by the *Goldberg-Coxeter construction* (extended below on 6-regular spheres) from Trifolium $\{1,3\}_1$.

Digression on Rose of Three Petals

- The polar equation of the rose (or *rhondonea*) is $r = a\cos n\theta$.
- Its case $n = 3$, Trifolium $\{1, 3\}_1$, is a *quartic plane curve*, i.e. a plane algebraic curve of degree 4, $r = \cos 3\theta$ in polar, or (using $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$) $(x^2 + y^2)^2 = x(x^2 - 3y^2)$ in rectangular coordinates.



Computer generation of the families

Main technique: exhaustive search. Sometimes, speedup by proving that a group of faces cannot be completed to the desired graph.

- The program CPF by Brinkmann-Delgado-Dress-Harmuth, 1997 generates 3-regular plane graphs with specified p-vector.
- ENU by Brinkmann-Harmuth-Heidemeier, 2003 and Heidemeier, 1998 does the same for 4-regular plane graphs. Dutour adapted ENU to deal with 2-gonal faces also.
- CGF by Harmuth generates 3-regular orientable maps with specified genus and p-vector.
- Plantri by Brinkmann-McKay deals with general graphs.
- The package CaGe by Brinkmann-Delgado-Dress-Harmuth, 1997 is used for plane graph drawings.
- The package PlanGraph by Dutour, 2002 is used for handling planar graphs in general.

II. Polyhedrality of $(\{a, b\}, k)$-spheres

Polyhedra and planar graphs

- A graph is called k-connected if after removing any set of $k-1$ vertices it remains connected.
- The skeleton of a polytope P is the graph $G(P)$ formed by its vertices, with two vertices adjacent if they generate a face.
- Steinitz Theorem: a graph is the skeleton of a polyhedron (3-polytope) if and only if it is planar and 3-connected.

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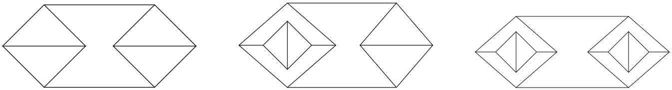
- A polyhedron is usually represented by the *Schlegel diagram* of its skeleton, the program used for this is CaGe.
- The dual graph G^* of a plane graph G is the plane graph formed by the faces of G, with two faces adjacent if they share an edge. The skeletons of dual polyhedra are dual.

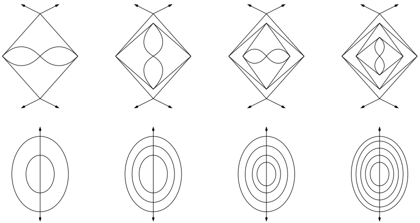
3-connectedness of $(\{a, b\}, 3)$-spheres

- Any $(\{a, b\}, k)$-sphere is 2-connected. But some infinite series of $(\{1, 2, 3\}, 6)$-spheres with $(p_1, p_2)=(2, 2)$ are *not*.
- Any $(\{a, 6\}, 3)$-sphere is 3-connected if $a = 4, 5$ and not if $a = 2$ (one can delete two vertices adjacent to a 2-gon).
- Except the following series, $(\{3, 6\}, 3)$-spheres (moreover, all $(\{3, 4, 5, 6\}, 3)$-spheres) are 3-connected.



3-connectedness of $(\{a, b\}, 6)$- and $(\{a, b\}, 4)$-spheres

- Any $(\{a, b\}, 6)$-sphere is 3-connected, except $(\{2, 3\}, 6)$- ones which are duals of only 2-connected $(\{3, 6\}, 3)$-spheres, with six vertices of degree 2 added on edges.
- Any $(\{a, b\}, 4)$-sphere is 3-connected, except the following series of $(\{2, 4\}, 4)$-spheres.



REMARK. $\{2, 4\}_v(D_{2d}, D_{2h})$ are *k-inflations* of above. D_4, D_{4h} are $GC_{k,l}(4 \times K_2)$. Remaining D_2: 2 complex or 3 natural parameters.

Hamiltonicity of $(\{a, b\}, k)$-spheres

- Grűnbaum-Zaks, 1974: all $(\{1,3\},6)$- and $(\{2,4\},4)$-spheres are Hamiltonian, but $(\{2,6\},3)$- with $v \equiv 0 \pmod 4$ are not
- Goodey, 1977: $(\{3,6\},3)$- and $(\{4,6\},3)$- are Hamiltonian.
- Conjecture: an Hamiltonian circuit exists in all other cases.

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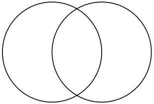
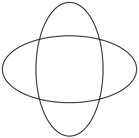
To check hamiltonicity of a $(\{a, b\}, k)$-map on the projective plane $\mathbb{P}^2$, the following theorem (Thomas-Yu, 1994) could help: every 4-connected graph on $\mathbb{P}^2$ has a *contractible* (i.e. being a boundary of 2-cell) Hamiltonian circuit.

III. 8 families:
4 smallest members

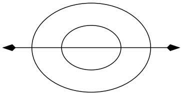
First four $(\{2,4\},4)$- and $(\{3,4\},4)$-spheres



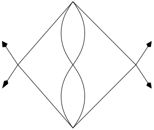
D_{4h} 2_1^2 (2^2)



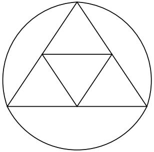
D_{4h} 4_1^2 (4^2)



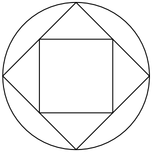
D_{2h} $2\times 2_1^2$ $(2^2, 4)$



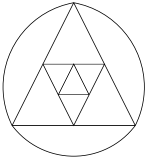
D_{2d} 6_2^2 (6^2)



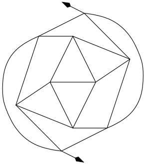
O_h 6_2^3 (4^3)
Borr. rings



D_{4d} 8_{18} (16)



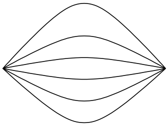
D_{3h} 9_{40} (18)



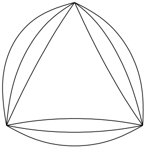
D_2 10_{56}^2
$(6; 14)$

Above links/knots are given in Rolfsen, 1976 and 1990 notation.

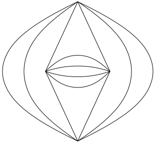
First four $(\{2,3\},6)$- and $(\{1,3\},6)$-spheres



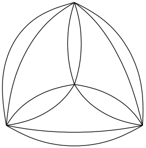
D_{6h} (2^3)



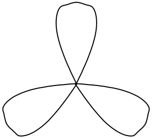
D_{3h} $(3;6)$



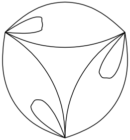
D_{2d} $(2^2;8)$



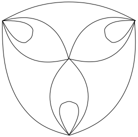
T_d (3^4)



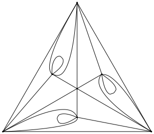
C_{3v} (3)



C_{3h} $(3;6)$



C_{3v} (6^2)

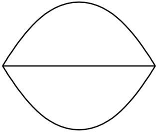


C_3 (21)

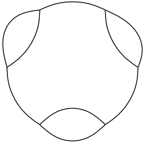
Grűnbaum-Zaks, 1974: $\{1,3\}_v$ exists iff $v = k^2 + kl + l^2$ for integers $0 \le l \le k$. We show that the number of $\{1,3\}_v$'s is the number of such representations of v, i.e. found $GC_{k,l}(\{1,3\}_1)$.

First four $(\{2,6\},3)$- and $(\{3,6\},3)$-spheres

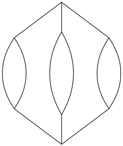
Number of $(\{2,6\}_v$'s is nr. of representations $v=2(k^2+kl+l^2)$, $0 \leq l \leq k$ $(GC_{k,l}(\{2,6\}_2))$. It become 2 for $v=7^2=5^2+15+3^2$.



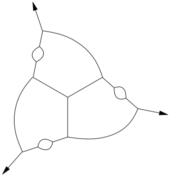
D_{3h} (6)



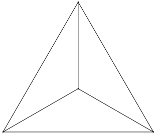
D_{3h} (6^3)



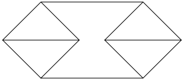
D_{3h} (12^2)



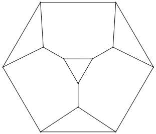
D_3 (42)



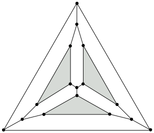
T_d (4^3)



D_{2h} $(8^2, 4^2)$

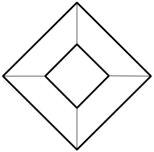


T_d (12^3)

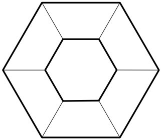


T_d (8^6)

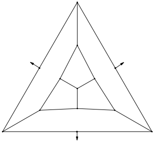
First four $(\{4,6\},3)$- and $(\{5,6\},3)$-spheres



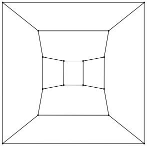
O_h (6^4)



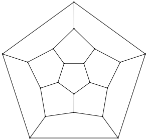
D_{6h} (18^2)



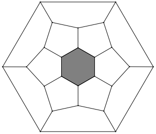
D_{3h} $(6^2; 30)$



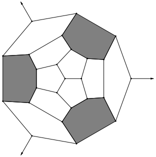
D_{2d} (24^2)



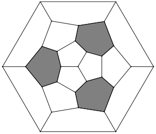
I_h (10^6)



D_{6d} $(12; 60)$



D_{3h} $(12^3; 42)$



T_d (12^7)

IV. Symmetry groups of $(\{a, b\}, k)$-spheres

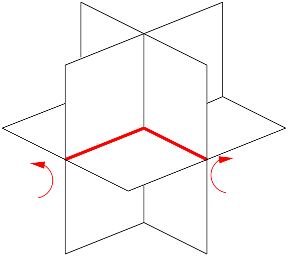
Finite isometry groups

All finite groups of isometries of 3-space $\mathbb{E}^3$ are classified.
In Schoenflies notations, they are:

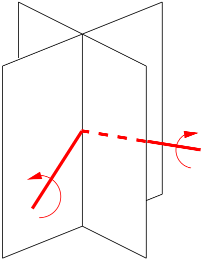
- C_1 is the trivial group
- C_s is the group generated by a plane reflexion
- $C_i = \{I_3, -I_3\}$ is the inversion group
- C_m is the group generated by a rotation of order m of axis Δ
- C_{mv} ($\simeq$ dihedral group) is the group generated by C_m and m reflexion containing Δ
- $C_{mh} = C_m \times C_s$ is the group generated by C_m and the symmetry by the plane orthogonal to Δ
- S_{2m} is the group of order $2m$ generated by an antirotation, i.e. commuting composition of a rotation and a plane symmetry

Finite isometry groups D_m, D_{mh}, D_{md}

- D_m ($\simeq$ dihedral group) is the group generated of C_m and m rotations of order 2 with axis orthogonal to Δ
- D_{mh} is the group generated by D_m and a plane symmetry orthogonal to Δ
- D_{md} is the group generated by D_m and m symmetry planes containing Δ and which does not contain axis of order 2



D_{2h}



D_{2d}

Remaining 7 finite isometry groups

- $I_h = H_3$ is the group of isometries of Dodecahedron; $I_h \simeq Alt_5 \times C_2$
- $I \simeq Alt_5$ is the group of rotations of Dodecahedron
- $O_h = B_3$ is the group of isometries of Cube
- $O \simeq Sym(4)$ is the group of rotations of Cube
- $T_d = A_3 \simeq Sym(4)$ is the group of isometries of Tetrahedron
- $T \simeq Alt(4)$ is the group of rotations of Tetrahedron
- $T_h = T \cup -T$

Remaining 7 finite isometry groups

- $I_h = H_3$ is the group of isometries of Dodecahedron; $I_h \simeq Alt_5 \times C_2$
- $I \simeq Alt_5$ is the group of rotations of Dodecahedron
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- $T_d = A_3 \simeq Sym(4)$ is the group of isometries of Tetrahedron
- $T \simeq Alt(4)$ is the group of rotations of Tetrahedron
- $T_h = T \cup -T$

While (point group) $Isom(P) \subset Aut(G(P))$ (combinatorial group), Mani, 1971: for any 3-polytope P, there is a 3-polytope P' with the same skeleton $G = G(P') = G(P)$, such that the group $Isom(P')$ of its isometries is isomorphic to $Aut(G)$.

8 families: symmetry groups

- 28 for $\{5,6\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}, S_4; C_3, C_{3v}, C_{3h}, S_6; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_5, D_{5h}, D_{5d}; D_6, D_{6h}, D_{6d}; T, T_d, T_h; I, I_h (Fowler-Manolopoulos, 1995)
- 16 for $\{4,6\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_6, D_{6h}; O, O_h (Deza-Dutour, 2005)
- 5 for $\{3,6\}_v$: D_2, D_{2h}, D_{2d}; T, T_d (Fowler-Cremona,1997)
- 2 for $\{2,6\}_v$: D_3, D_{3h} (Grűnbaum-Zaks, 1974)
- 18 for $\{3,4\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}, S_4; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_4, D_{4h}, D_{4d}; O, O_h (Deza-Dutour-Shtogrin, 2003)
- 5 for $\{2,4\}_v$: D_2, D_{2h}, D_{2d}; D_4, D_{4h}, all in $[D_2, D_{4h}]$ (same)
- 3 for $\{1,3\}_v$: C_3, C_{3v}, C_{3h} (Deza-Dutour, 2010)
- 22 for $\{2,3\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}, S_4; C_3, C_{3v}, C_{3h}, S_6; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_6, D_{6h}; T, T_d, T_h (same)

8 families: Goldberg-Coxeter construction $GC_{k,l}(.)$

Agregating groups $\mathbf{C_1}$=$\{C_1, C_s, C_i\}$, $\mathbf{C_m}$=$\{C_m, C_{mv}, C_{mh}, S_{2m}\}$, $\mathbf{D_m}$=$\{D_m, D_{mh}, D_{md}\}$, and $\mathbf{T}$=$\{T, T_d, T_h\}$, we get

- for $\{5,6\}_v$: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_5}$, $\mathbf{D_6}$, $\mathbf{T}$, $\{I, I_h\}$
- for $\{2,3\}_v$: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, $\mathbf{T}$
- for $\{4,6\}_v$: $\mathbf{C_1}$, $\mathbf{C_2}\backslash S_4$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, $\{O, O_h\}$
- for $\{3,4\}_v$: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\{O, O_h\}$
- for $\{3,6\}_v$: $\mathbf{D_2}$, $\{T, T_d\}$
- for $\{2,4\}_v$: $\mathbf{D_2}$, $\{D_4, D_{4h}\}$
- for $\{2,6\}_v$: $\{D_3, D_{3h}\}$
- for $\{1,3\}_v$: $\mathbf{C_3}\backslash S_6$=$\{C_3, C_{3v}, C_{3h}\}$

8 families: Goldberg-Coxeter construction $GC_{k,l}(.)$

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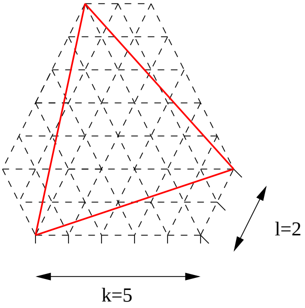
- for $\{5,6\}_v$: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_5}$, $\mathbf{D_6}$, $\mathbf{T}$, $\{I, I_h\}$
- for $\{2,3\}_v$: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, $\mathbf{T}$
- for $\{4,6\}_v$: $\mathbf{C_1}$, $\mathbf{C_2}\backslash S_4$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, $\{O, O_h\}$
- for $\{3,4\}_v$: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\{O, O_h\}$
- for $\{3,6\}_v$: $\mathbf{D_2}$, $\{T, T_d\}$
- for $\{2,4\}_v$: $\mathbf{D_2}$, $\{D_4, D_{4h}\}$
- for $\{2,6\}_v$: $\{D_3, D_{3h}\}$
- for $\{1,3\}_v$: $\mathbf{C_3}\backslash S_6$=$\{C_3, C_{3v}, C_{3h}\}$

Spheres of blue symmetry are $GC_{k,l}$ from 1st such; so, given by one complex (gaussian for k=4, Eisenstein for k=3,6) parameter. Goldberg, 1937 and Coxeter, 1971: $\{5,6\}_v(I, I_h)$, $\{4,6\}_v(O, O_h)$, $\{3,6\}_v(T, T_d)$. Dutour-Deza, 2004 and 2010: for other cases.

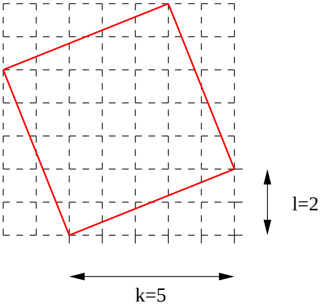
V. Goldberg-Coxeter construction

Goldberg-Coxeter construction $GC_{k,l}(.)$

- Take a 3- or 4-regular plane graph G. The faces of dual graph G^* are triangles or squares, respectively.
- Break each face into pieces according to parameter (k, l). *Master polygons* below have area $\mathcal{A}(k^2+kl+l^2)$ or $\mathcal{A}(k^2+l^2)$, where $\mathcal{A}$ is the area of a small polygon.



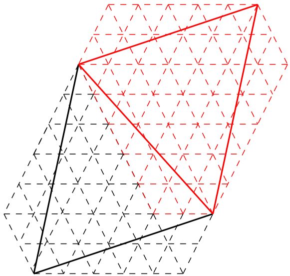
3−valent case



4−valent case

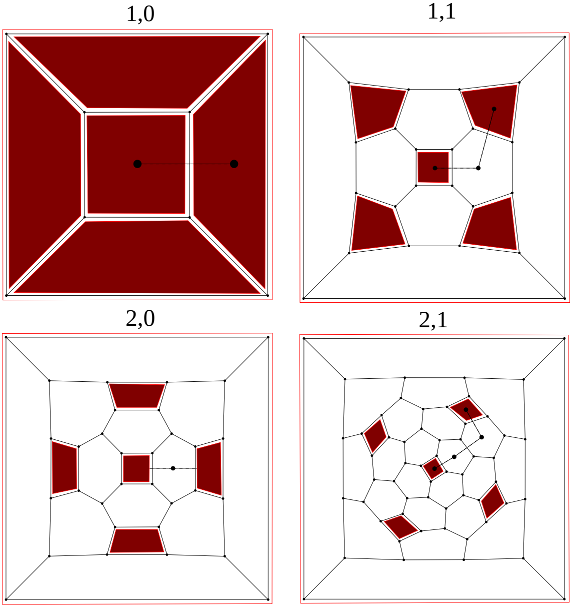
Gluing the pieces together in a coherent way

- Gluing the pieces so that, say, 2 non-triangles, coming from subdivision of neighboring triangles, form a small triangle, we obtain another triangulation or quadrangulation of the plane.

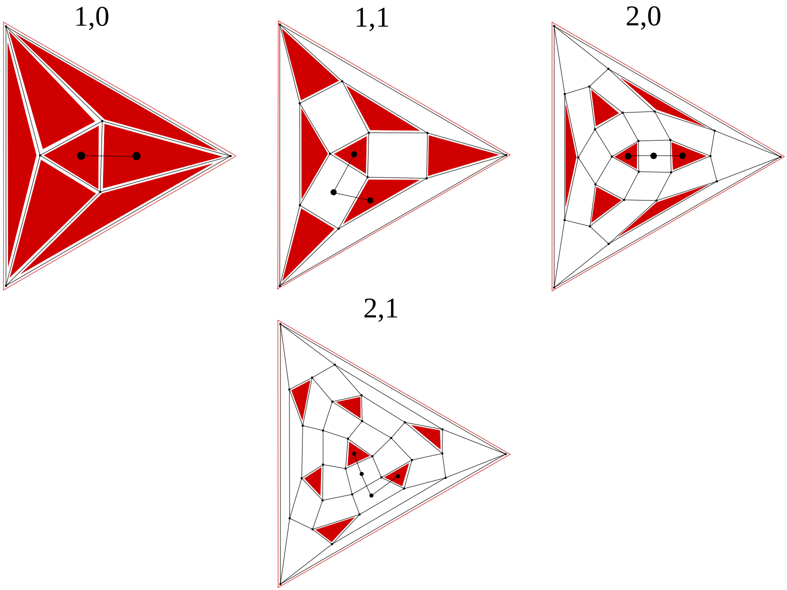


- The dual is a 3- or 4-regular plane graph, denoted $GC_{k,l}(G)$; we call it Goldberg-Coxeter construction.
- It works for any 3- or 4-regular map on oriented surface.

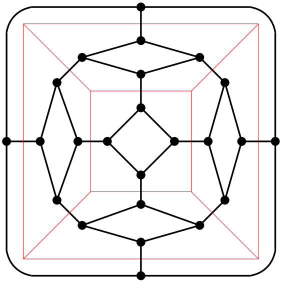
$GC_{k,l}(Cube)$ for $(k,l) = (1,0), (1,1), (2,0), (2,1)$



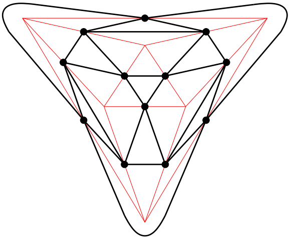
Goldberg-Coxeter construction from Octahedron



The case $(k, l) = (1, 1)$



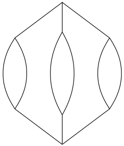
3-regular case
$GC_{1,1}$ is called leapfrog
($\frac{1}{3}$-truncation of the dual)
truncated Octahedron



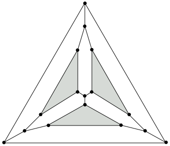
4-regular case
$GC_{1,1}$ is called medial
($\frac{1}{2}$-truncation)
Cuboctahedron

The case $(k, l) = (k, 0)$ of $GC_{k,l}(G)$: k-inflation

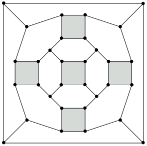
Chamfering (*quadrupling*) $GC_{2,0}(G)$ of 8 1st $(\{a, b\}, k)$-spheres, (a, b)=$(2, 6), (3, 6), (4, 6), (5, 6)$ and $(2, 4), (3, 4), (1, 3), (2, 3)$, are:



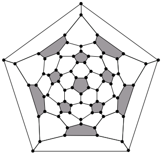
D_{3h} (12^2)



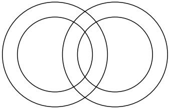
T_d (8^6)



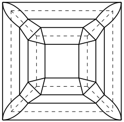
O_h (12^8)



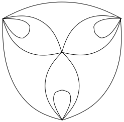
I_h (20^{12})



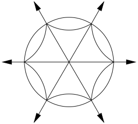
D_{4h} (4^4)



O_h (8^6)



C_{3v} (6^2)

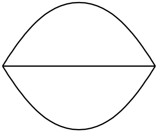


D_{6h} $(4^3, 6^2)$

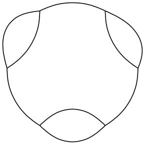
For 4-regular G, $GC_{2k^2,0}(G) = GC_{k,k}(GC_{k,k}(G))$ by $(k+ki)^2 = 2k^2 i$.

First four $GC_{k,l}(3 \times K_2)$ and $GC_{k,l}(4 \times K_2)$

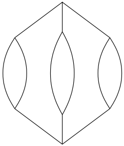
All $(\{2,6\},3)$-spheres are $G_{k,l}(3{\times}K_2)$: D_{3h}, D_{3h}, D_3 if $l{=}0, k$, else.



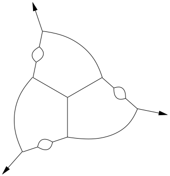
D_{3h} $3 \times K_2$



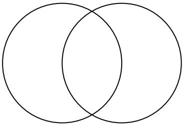
D_{3h} leapfrog



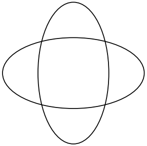
D_{3h} $G_{2,0}$



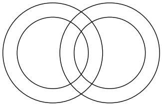
D_3 $G_{2,1}$



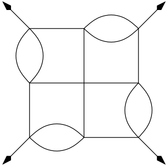
D_{4h} $4 \times K_2$



D_{4h} medial

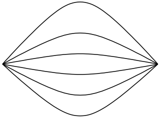


D_{4h} $G_{2,0}$

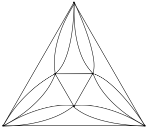


D_4 $G_{2,1}$

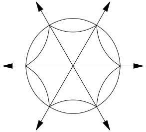
First four $GC_{k,l}(6\times K_2)$ and $GC_{k,l}(Trifolium)$



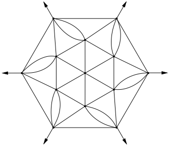
D_{6h}



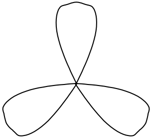
D_{3d} $G_{1,1}$



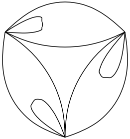
D_{6h} $G_{2,0}$



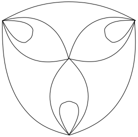
D_6 $G_{2,1}$



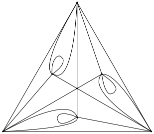
C_{3v}



C_{3h} $G_{1,1}$



C_{3v} $G_{2,0}$



C_3 $G_{2,1}$

All $(\{2,3\},6)$-spheres are $G_{k,l}(6\times K_2)$: C_{3v}, C_{3h}, C_3 if $l=0$, k, else.

Plane tilings $\{4^4\}$, $\{3^6\}$ and complex rings $\mathbb{Z}[i]$, $\mathbb{Z}[w]$

- The vertices of regular plane tilings $\{4^4\}$ and $\{3^6\}$ form each, convenient algebraic structures: lattice and ring. Path-metrics of those graphs are l_1- 4-*metric* and *hexagonal* 6-*metric*.
- $\{4^4\}$: square lattice $\mathbb{Z}^2$ and ring $\mathbb{Z}[i]=\{z=k+li : k,l \in \mathbb{Z}\}$ of gaussian integers with norm $N(z)=z\overline{z}=k^2+l^2=||(k,l)||^2$.
- $\{3,6\}$: hexagonal lattice $A^2=\{x \in \mathbb{Z}^3 : x_0+x_1+x_2=0\}$ and ring $\mathbb{Z}[w]=\{z=k+lw : k,l \in \mathbb{Z}\}$, where $w=e^{i\frac{\pi}{3}}=\frac{1}{2}(1+i\sqrt{3})$, of Eisenstein integers with norm $N(z)=z\overline{z}=k^2+kl+l^2=\frac{1}{2}||x||^2$ We identify points $x=(x_0,x_1,x_2) \in A^2$ with $x_0+x_1 w \in \mathbb{Z}[w]$.

Plane tilings $\{4^4\}$, $\{3^6\}$ and complex rings $\mathbb{Z}[i]$, $\mathbb{Z}[w]$

- The vertices of regular plane tilings $\{4^4\}$ and $\{3^6\}$ form each, convenient algebraic structures: lattice and ring. Path-metrics of those graphs are l_1- 4-*metric* and *hexagonal* 6-*metric*.
- $\{4^4\}$: square lattice $\mathbb{Z}^2$ and ring $\mathbb{Z}[i]=\{z=k+li : k,l\in\mathbb{Z}\}$ of gaussian integers with norm $N(z)=z\overline{z}=k^2+l^2=||(k,l)||^2$.
- $\{3,6\}$: hexagonal lattice $A^2=\{x\in\mathbb{Z}^3 : x_0+x_1+x_2=0\}$ and ring $\mathbb{Z}[w]=\{z=k+lw : k,l\in\mathbb{Z}\}$, where $w=e^{i\frac{\pi}{3}}=\frac{1}{2}(1+i\sqrt{3})$, of Eisenstein integers with norm $N(z)=z\overline{z}=k^2+kl+l^2=\frac{1}{2}||x||^2$ We identify points $x=(x_0,x_1,x_2)\in A^2$ with $x_0+x_1w\in\mathbb{Z}[w]$.
- A natural number $n=\prod_i p_i^{\alpha_i}$ is of form $n=k^2+l^2$ if and only if any α_i is even, whenever $p_i\equiv 3(mod\, 4)$ (*Fermat Theorem*). It is of form $n=k^2+kl+l^2$ if and only if $p_i\equiv 2 \pmod 3$.
- The first cases of non-unicity with $gcd(k,l)=gcd(k_1,l_1)=1$ are $91=9^2+9+1^2=6^2+30+5^2$ and $65=8^2+1^2=7^2+4^2$. The first cases with $l=0$ are $7^2=5^2+15+3^2$ and $5^2=4^2+3^2$.

The bilattice of vertices of hexagonal plane tiling $\{6^3\}$

- Let us identify the *hexagonal lattice* A^2 (or *equilateral triangular lattice* of the vertices of the *regular plane tiling* $\{3^6\}$) with *Eisenstein ring* (of Eisenstein integers) $\mathbb{Z}[w]$.
- The hexagon centers of $\{6^3\}$ form $\{3^6\}$. Also, with vertices of $\{6^3\}$, they form $\{3^6\}$, rotated by 90° and scaled by $\frac{1}{3}\sqrt{3}$.
- The complex coordinates of vertices of $\{6^3\}$ are given by vectors $v_1{=}1$ and $v_2{=}w$. The lattice $L{=}\mathbb{Z}v_1{+}\mathbb{Z}v_2$ is $\mathbb{Z}[w]$.
- The vertices of $\{6^3\}$ form bilattice $L_1 \cup L_2$, where the bipartite complements, $L_1{=}(1{+}w)L$ and $L_2{=}1{+}(1{+}w)L$, are stable under multiplication. Using this,

$GC_{k,l}(G)$ for 6-regular graph G can be defined similarly to 3- and 4-regular case, but only for $k + lw \in L_2$, i.e. $k \equiv l \pm 1\,(mod\,3)$.

Ring formalism

$\mathbb{Z}[i]$ (gaussian integers) and $\mathbb{Z}[\omega]$ (Eisenstein integers) are *unique factorization* rings

Dictionary

	3-regular G	4-regular G	6-regular G
the ring	Eisenstein $\mathbb{Z}[\omega]$	gaussian $\mathbb{Z}[i]$	Eisenstein $\mathbb{Z}[\omega]$
Euler formula	$\sum_i (6-i)p_i = 12$	$\sum_i (4-i)p_i = 8$	$\sum_i (3-i)p_i = 6$
curvature 0	hexagons	squares	triangles
ZC-circuits	zigzags	central circuits	both
$GC_{11}(G)$	leapfrog graph	medial graph	or. tripling

Goldberg-Coxeter operation in ring terms

- Associate $z=k+lw$ (Eisenstein) or $z=k+li$ (gaussian integer) to the pair (k, l) in 3-,6- or 4-regular case. Operation $GC_z(G)$ correspond to scalar multiplication by $z=k+lw$ or $k+li$.
- Writing $GC_z(G)$, instead of $GC_{k,l}(G)$, one has:

$$GC_z(GC_{z'}(G)) \quad = \quad GC_{zz'}(G)$$

- If G has v vertices, then $GC_{k,l}(G)$ has $vN(z)$ vertices, i.e., $v(k^2+l^2)$ in 4-regular and $v(k^2+kl+l^2)$ in 3- or 6-reg. case.

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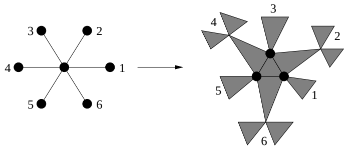
- If G has v vertices, then $GC_{k,l}(G)$ has $vN(z)$ vertices, i.e., $v(k^2{+}l^2)$ in 4-regular and $v(k^2{+}kl{+}l^2)$ in 3- or 6-reg. case.
- $GC_z(G)$ has all rotational symmetries of G in 3- and 4-regular case, and all symmetries if $l{=}0,k$ in general case.
- $GC_z(G){=}GC_{\overline{z}}(\overline{G})$ where $\overline{G}$ differs by a plane symmetry only from G. So, if G has a symmetry plane, we reduce to $0{\leq}l{\leq}k$; otherwise, graphs $GC_{k,l}(G)$ and $GC_{l,k}(G)$ are not isomorphic.

$GC_{k,l}(G)$ for 6-regular plane graph G and *any* k, l

- Bipartition of G^* gives vertex 2-coloring, say, red/blue of G.
- Truncation $Tr(G)$ of $\{1,2,3\}_v$ is a 3-regular $\{2,4,6\}_{6v}$.
- Coloring white vertices of G gives face 3-coloring of $Tr(G)$. White faces in $Tr(G)$ correspond to such in $GC_{k,l}(Tr(G))$.
- For $k \equiv l \pm 1\,(mod\,3)$, i.e. $k + lw \in L_2$, define $GC_{k,l}(G)$ as $GC_{k.l}(Tr(G))$ with all white faces shrinked.
- If $k \equiv l\,((mod\,3)$, faces of $Tr(G)$ are white in $GC_{k,l}(Tr(G))$. Among 3 faces around each vertex, one is white. Coloring other red gives unique 3-coloring of $GC_{k,l}(Tr(G))$. Define $GC_{k,l}(G)$ as pair G_1, G_2 with $Tr(G_1){=}Tr(G_2){=}GC_{k,l}(Tr(G))$ obtained from it by shrinking all red or blue faces.
- $GC_{1,0}(G) = G$ and $GC_{1,1}(G)$ is oriented tripling.

Oriented tripling $GC_{1,1}(G)$ of 6-regular plane graph G

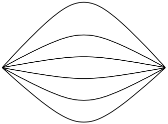
- Let C_1, C_2 be bipartite classes of G^*. For each C_i, oriented tripling $Or_{C_i}(G)$ (or $GC_{1,1}(G)$) is 6-regular plane graph coming by vertex of $G \rightarrow 3$ vertices and 4 triangular faces of $Or_{C_i}(G)$. Symmetries of $Or_{C_i}(G)$ are symmetries of G preserving C_i.
- Orient edges of C_i clockwise. Select 3 of 6 neighbors of each vertex v: $\{2,4,6\}$ are those with directed edge going to v; for $\{1,5,5\}$, edges go to them.



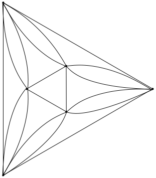
- Any $z{=}k{+}lw{\neq}0$ with $k{\equiv}l \pmod 3$ can be written as $(1{+}w)^s(k'{+}l'w)w$, where $s{\geq}0$ and $k'{\equiv}l' \pm 1 \pmod 3$. So, $GC_{k,l}(G){=}G_{k',l'}(Or^s(G))$.

Examples of oriented tripling $GC_{1,1}(G)$

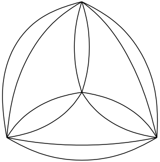
Below: $\{2,3\}_2$ and $\{2,3\}_4$ have unique oriented tripling.



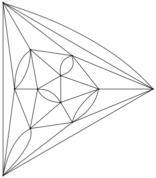
2 D_{6h}



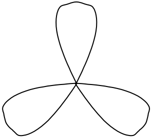
6 D_{3d}



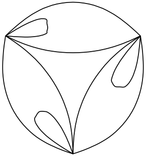
4 T_d



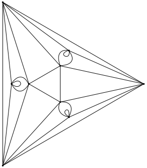
12 T_h



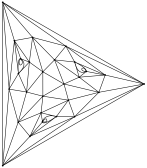
1 C_{3v}



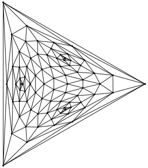
3 C_{3h}



9 C_{3v}



27 C_{3h}

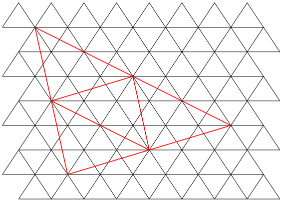


81 C_{3v}

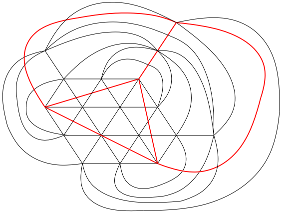
Above: first 4 consecutive orient triplings of the Trifolium.

VI. Parameterizing $(\{a, b\}, k)$-spheres

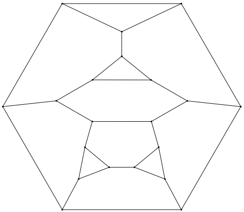
Example: construction of the $(\{3,6\},3)$-spheres in $Z[\omega]$



In the central triangle ABC, let A be the origin of the complex plane



The corresponding triangulation



All $(\{3,6\},3)$-spheres come this way; two complex parameters in $Z[\omega]$ defined by the points B and C

Parameterizing $(\{a, b\}, k)$-spheres

Thurston, 1998 implies: $(\{a, b\}, k)$-spheres have p_a-2 parameters and the number of v-vertex ones is $O(v^{m-1})$ if $m{=}p_a\text{-}2 > 2$.
Idea: since b-gons are of zero curvature, it suffices to give relative positions of a-gons having curvature $2k - a(k-2) > 0$.
At most $p_a - 1$ vectors will do, since one position can be taken 0.
But once $p_a - 1$ a-gons are specified, the last one is constrained.
The number of m-parametrized spheres with at most v vertices is $O(v^m)$ by direct integration. The number of such v-vertex spheres is $O(v^{m-1})$ if $m > 1$, by a *Tauberian* theorem.

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- Goldberg, 1937: $\{a, 6\}_v$ (highest 2 symmetries): 1 parameter
- Fowler and al., 1988: $\{5, 6\}_v$ (D_5, D_6 or T): 2 parameters.
- Grűnbaum-Motzkin, 1963: $\{3, 6\}_v$: 2 parameters.
- Deza-Shtogrin, 2003: $\{2, 4\}_v$; 2 parameters.
- Thurston, 1998: $\{5, 6\}_v$: 10 (again complex) parameters.
  Graver, 1999: $\{5, 6\}_v$: 20 integer parameters.

8 families: number of complex parameters by groups

- $\{5,6\}_v$ $\mathbf{C_1}(10)$, $\mathbf{C_2}(6)$, $\mathbf{C_3}(4)$, $\mathbf{D_2}(4)$, $\mathbf{D_3}(3)$, $\mathbf{D_5}(2)$, $\mathbf{D_6}(2)$, $\mathbf{T}(2)$, $\{I, I_h\}(1)$
- $\{4,6\}_v$ $\mathbf{C_1}(4)$, $\mathbf{C_2}\backslash S_4(3)$, $\mathbf{D_2}(2)$, $\mathbf{D_3}(2)$, $\{D_6, D_{6h}\}(1)$, $\{O, O_h\}(1)$
- $\{3,4\}_v$ $\mathbf{C_1}(6)$, $\mathbf{C_2}(4)$, $\mathbf{D_2}(3)$, $\mathbf{D_3}(2)$, $\mathbf{D_4}(2)$, $\{O, O_h\}(1)$
- $\{2,3\}_v$ $\mathbf{C_1}(4)$, $\mathbf{C_2}(3?)$, $\mathbf{C_3}(3?)$, $\mathbf{D_2}(2?)$, $\mathbf{D_3}(2?)$, $\mathbf{T}(1)$, $\{D_6, D_{6h}\}(1)$
- $\{3,6\}_v$ $\mathbf{D_2}(2)$, $\{T, T_d\}(1)$
- $\{2,4\}_v$ $\mathbf{D_2}(2)$, $\{D_4, D_{4h}\}(1)$
- $\{2,6\}_v$ $\{D_3, D_{3h}\}(1)$
- $\{1,3\}_v$ $\{C_3, C_{3v}, C_{3h}\}(1)$

Thurston, 1998 implies: $(\{a,b\},k)$-spheres have p_a-2 parameters and the number of v-vertex ones is $O(v^{m-1})$ if $m{=}p_a\text{-}2 > 1$.

Number of complex parameters

$\{5,6\}_v$

Group	$\#param.$
$\mathbf{C_1}$	10
$\mathbf{C_2}$	6
$\mathbf{C_3}, \mathbf{D_2}$	4
$\mathbf{D_3}$	3
$\mathbf{D_5}, \mathbf{D_6}, \mathbf{T}$	2
$\mathbf{I}$	1

$\{3,4\}_v$

Group	$\#param.$
$\mathbf{C_1}$	6
$\mathbf{C_2}$	4
$\mathbf{D_2}$	3
$\mathbf{D_3}, \mathbf{D_4}$	2
$\mathbf{O}$	1

$\{4,6\}_v$

Group	$\#param.$
$\mathbf{C_1}$	4
$\mathbf{C_2}$	3
$\mathbf{D_2}, \mathbf{D_3}$	2
$\mathbf{D_6}, \mathbf{O}$	1

$\{2,3\}_v$

Group	$\#param.$
C_1	4
$\mathbf{C_2}, \mathbf{C_3}$	3?
$\mathbf{D_2}, \mathbf{D_3}$	2?
$\mathbf{D_6}, \mathbf{T}$	1

$\{3,6\}_v$- and $\{2,4\}_v$: 2 complex parameters but 3 natural ones will do: *pseudoroad* length, number of circumscribing *railroads*, *shift*.

VII. Railroads and tight $(\{a, b\}, k)$-spheres

ZC-circuits

- The edges of any plane graph are doubly covered by zigzags (Petri or left-right paths), i.e., circuits such that any two but not three consecutive edges bound the same face.
- The edges of any *Eulerian* (i.e., even-valent) plane graph are partitioned by its central circuits (those going straight ahead).
- A ZC-circuit means *zigzag* or *central circuit* as needed.
  CC- or Z-vector enumerate lengths of above circuits.

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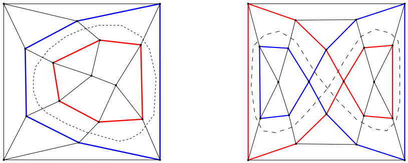
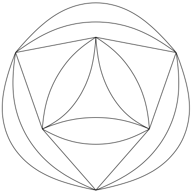
- A ZC-circuit means *zigzag* or *central circuit* as needed. CC- or Z-vector enumerate lengths of above circuits.
- A railroad in a 3-, 4- or 6-regular plane graph is a circuit of 6-, 4- or 3-gons, each adjacent to neighbors on opposite edges. Any railroad is bound by two "parallel" ZC-circuits. It (any if 4-, simple if 3- or 6-regular) can be collapsed into 1 ZC-circuit.

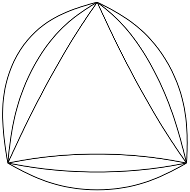


Railroad in a 6-regular sphere: examples

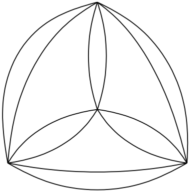
$APrism_3$ with 2 base 3-gons doubled is the $\{2,3\}_6$ (D_{3d}) with CC-vector $(3^2, 4^3)$, all five central circuits are simple.
Base 3-gons are separated by a simple railroad R of six 3-gons, bounded by two parallel central 3-circuits around them. Collapsing R into one 3-circuit gives the $\{2,3\}_3$ (D_{3h}) with CC-vector $(3; 6)$.



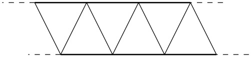
D_{3d} $(3^2, 4^3)$



D_{3h} $(3; 6)$



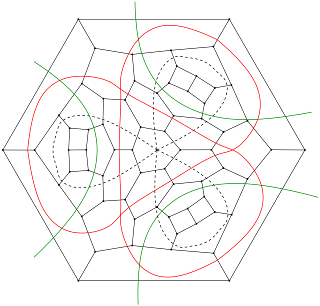
T_d (3^4)



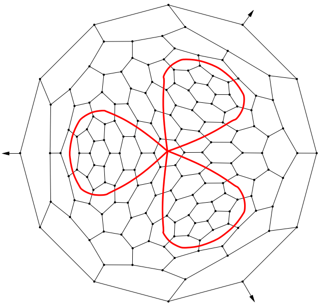
Above $\{2,3\}_4$ (T_d) has no railroads but it is not strictly tight, i.e. no any central circut is adjacent to a non-3-gon *on each side*.

Railroads flower: Trifolium $\{1,3\}_1$

Railroads can be simple or self-intersect, including triply if $k=3$.
First such Dutour $(\{a,b\},k)$-spheres for $(a,b)=(4,6),(5,6)$ are:



$\{4,6\}_{66}(D_{3h})$ twice



$\{5,6\}_{172}(C_{3v})$

Which plane curves with at most triple self-intersectionss come so?

Number of ZC-circuits in tight $(\{a, b\}, k)$-sphere

- Call an $(\{a, b\}, k)$-sphere tight if it has no railroads.

- ≤ 15 for $\{5, 6\}_v$ Dutour, 2004
- ≤ 9 for $\{4, 6\}_v$ and $\{2, 3\}_v$ Deza-Dutour, 2005 and 2010
- ≤ 3 for $\{2, 6\}_v$ and $\{1, 3\}_v$ same
- ≤ 6 for $\{3, 4\}_v$ Deza-Shtogrin, 2003
- Any $\{3, 6\}_v$ has ≥ 3 zigzags with equality iff it is tight. All $\{3, 6\}_v$ are tight iff $\frac{v}{4}$ is prime and none iff it is even.
- Any $\{2, 4\}_v$ has ≥ 2 central circuits with equality iff it is tight. There is a tight one for any even v.

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- Any $\{2, 4\}_v$ has ≥ 2 central circuits with equality iff it is tight. There is a tight one for any even v.

First tight ones with max. of ZC-circuits are $GC_{21}(\{a, b\}_{\min})$: $\{5, 6\}_{140}(I)$, $\{4, 6\}_{56}(O)$, $\{2, 6\}_{14}(D_3)$, $\{3, 4\}_{30}(0)$; $\{2, 3\}_{44}(D_{3h})$ and $\{a, b\}_{\min}$: $\{3, 6\}_4(T_d)$, $\{2, 4\}_2(D_{4h})$. Besides $\{2, 3\}_{44}(D_{3h})$, ZC-circuits are: $(28^{15}), (21^8), (14^3), (10^6), (4^3), (2^2)$, all simple.

Maximal number M_v of central circuits in *any* $\{2,3\}_v$

- $M_v = \frac{v}{2}+1, \frac{v}{2}+2$ for $v \equiv 0, 2 \pmod 4$. It is realized by the series of symmetry D_{2d} with CC-vector $2^{\frac{v}{2}}, 2v_{0,v}$ and of symmetry D_{2h} with CC-vector $2^{\frac{v}{2}}, v^2_{0,\frac{v-2}{4}}$ if $v \equiv 0, 2 \pmod 4$.
- For odd v, M_v is $\lfloor \frac{v}{3} \rfloor + 3$ if $v \equiv 2, 4, 6 \pmod 9$ and $\lfloor \frac{v}{3} \rfloor + 1$, otherwise. Define t_v by $\frac{v-t_v}{3} = \lfloor \frac{v}{3} \rfloor$. M_v is realized by the series of symmetry C_{3v} if $v \equiv 1 \pmod 3$ and D_{3h}, otherwise. CC-vector is $3^{\lfloor \frac{v}{3} \rfloor}, (2\lfloor \frac{v}{3} \rfloor + t_v)^3_{0,\lfloor \frac{v-2t_v}{9} \rfloor}$ if $v \equiv 2, 4, 6 \pmod 9$ and $3^{\lfloor \frac{v}{3} \rfloor}, (2v + t_v)_{0,v+2t_v}$, otherwise.
- The minimal number of central circuits, 1, have c-knotted $\{2,3\}_v$. They correspond to (*some of*) plane curves with only triple self-intersection points. For $v = 4, \dots, 14, 15$, their number is $1, 0, 2, 0, 2, 0, 2, 0, 4, 0, 11, 9, 1$..

VIII. Tight pure $(\{a,b\},k)$-spheres

Tight $(\{a, b\}, k)$-spheres with only simple ZC-circuits

- Call $(\{a, b\}, k)$-sphere pure if any of its ZC-circuits is *simple*, i.e. has no self-intersections. Such ZC-circuit can be seen as a Jordan curve, i.e. a plane curve which is topologically equivalent to (a homeomorphic image of) the unit circle.
- Any $(\{3, 6\}, 3)$- or $(\{2, 4\}, 4)$-sphere is pure. They are tight if and only if have three or, respectively, two ZC-circuits.
- Any ZC-circuit of $\{2, 6\}_v$ or $\{1, 3\}_v$ self-intersects.

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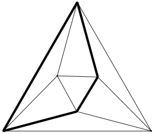
- Any ZC-circuit of $\{2, 6\}_v$ or $\{1, 3\}_v$ self-intersects.

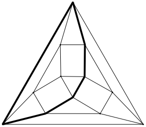
The number of tight pure $(\{a, b\}, k)$-spheres is:

- 9? for $\{5, 6\}_v$ computer-checked for $v \leq 300$ by Brinkmann
- 2 for $\{4, 6\}_v$ Deza-Dutour, 2005
- 8 for $\{3, 4\}_v$ same
- 5 for $\{2, 3\}_v$ same, 2010

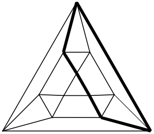
All tight $(\{3,4\},4)$-spheres with only simple central circuits



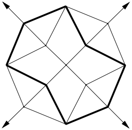
6 O_h (4^3) *Octahedron*



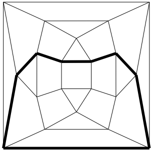
12 O_h (6^4) $GC_{11}(Oct.)$



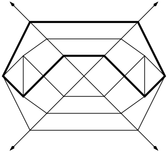
12 D_{3h} (6^4)



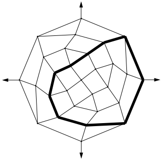
14 D_{4h} $(6^2, 8^2)$



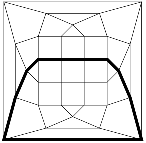
20 D_{2d} (8^5)



22 D_{2h} $(8^3, 10^2)$



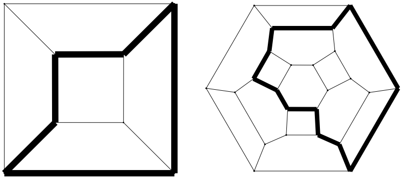
30 O (10^6) $GC_{21}(Oct.)$



32 D_{4h} $(10^4, 12^2)$

All tight $(\{4,6\},3)$-spheres with only simple zigzags

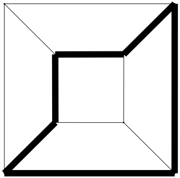
There are exactly two such spheres: Cube and its leapfrog $GC_{11}(Cube)$, truncated Octahedron.



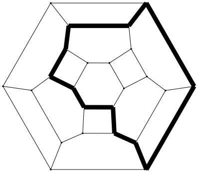
6 O_h (6^4) **24** O_h (10^6)

All tight $(\{4,6\},3)$-spheres with only simple zigzags

There are exactly two such spheres: Cube and its leapfrog $GC_{11}(Cube)$, truncated Octahedron.



6 O_h (6^4)

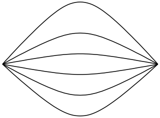


24 O_h (10^6)

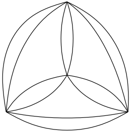
Proof is based on a) The size of intersection of two simple zigzags in any $(\{4,6\},3)$-sphere is $0,2,4$ or 6 and
b) Tight $(\{4,6\},3)$-sphere has at most 9 zigzags.

For $(\{2,3\},6)$-spheres, a) holds also, implying a similar result.

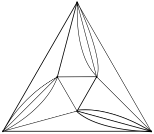
Tight $(\{2,3\},6)$-spheres with only simple ZC-circuits



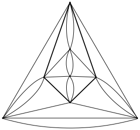
2 D_{6h} (2^3)
6^2



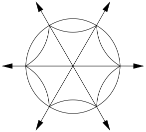
4 T_d (3^4)
6^4



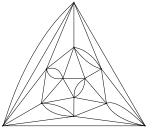
6 D_3 no
$12, 8^3$



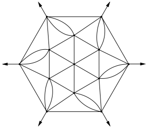
8 D_{2d} $(5^4, 4)$
no



D_{6h} $(4^3, 6^2)$
8 8^6 no



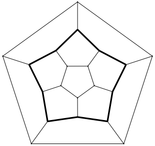
12 T_h (6^6)
12^6



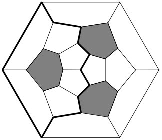
14 D_6 no
14^6

All pure CC-tight: Nrs. 1,2,4,5,6. All pure Z-tight: Nrs. 1,2,3,6,7.
1st, 3rd are strictly CC-, Z-tight: all ZC-circuits sides touch 2-gons.

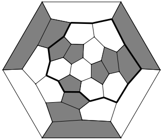
7 tight $(\{5,6\},3)$-spheres with only simple zigzags



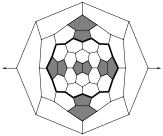
20 I_h (10^6)



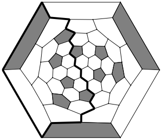
28 T_d (12^7)



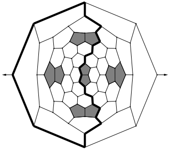
48 D_3 (16^9)



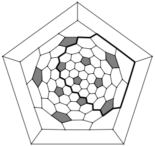
76 D_{2d}
$(22^4, 20^7)$



88 T (22^{12})



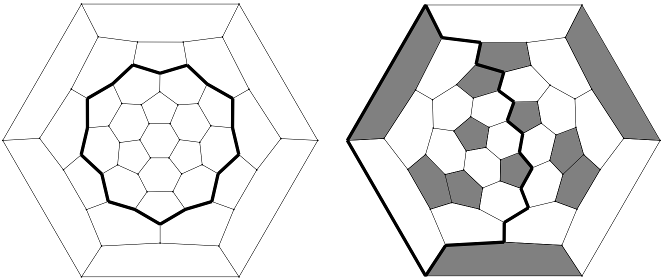
92 T_h
$(24^6, 22^6)$



140 I, (28^{15})

The zigzags of $1,2,3,5,7$th above and next two form 7 Grűnbaum arrangements of Jordan curves, i.e. any two intersect in 2 points. The groups of $1,5,7$th and $\{5,6\}_{60}(I_h)$ are zigzag-transitive.

Two other such $(\{5,6\},3)$-spheres



60 I_h (18^{10}) **60** D_3 (18^{10})

This pair was first answer on a question in Grűnbaum, 1967, 2003 *Convex Polytopes* about existence of *simple* polyhedra with the same p-vector but different zigzags. The groups of above $\{5,6\}_{60}$ have, acting on zigzags, 1 and 3 orbits, respectively.

IX. Infinite families of $(\{a, b\}, k)$-maps on surfaces

Non-hyperbolic (R, k)-maps

- Given $R \subset \mathbb{N}$ and a surface $\mathbb{F}^2$, an (R, k)-$\mathbb{F}^2$ is a k-regular map M on $\mathbb{F}^2$ whose faces have gonalities $i \in R$.
- Again, let our maps be non-hyperbolic, i.e., $\frac{1}{k} + \frac{1}{m} \geq \frac{1}{2}$ for $m = max\{i \in R\}$. So, it holds $m \leq \frac{2k}{k-2}$.
- Euler characteristic $\chi(M)$ is $v - e + f$, where v, e and $f = \sum_i p_i$ are the numbers of vertices, edges and faces of M.
- Since k-regularity implies $kv = 2e = \sum_i ip_i$, Euler formula $\chi = v - e + f$ becomes $2\chi(M)k = \sum_i p_i(2k - i(k-2))$.

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- The family of (R,k)-maps can be infinite only if $m = \frac{2k}{k-2}$ (i.e., for parabolic maps), when p_m is not restricted.
- Also, $\chi \geq 0$ with $\chi = 0$ if and only if $R = \{m\}$; and all possible pairs (m,k) are $(6,3)$, $(4,4)$, $(3,6)$.
- $(\{a,b\},k)$-maps have $b=\frac{2k}{k-2}$, $p_a=\frac{\chi b}{b-a}$ and $v=\frac{1}{k}(ap_a + bp_b)$.

The $(\{a, b\}, k)$-maps on torus and Klein bottle

The compact closed (i.e. without boundary) irreducible surfaces are: sphere $\mathbb{S}^2$, torus $\mathbb{T}^2$ (two orientable), real projective (elliptic) plane $\mathbb{P}^2$ and Klein bottle $\mathbb{K}^2$ with $\chi = 2, 0, 0, 1$, respectively.

The maps $(\{a, b\}, k)$-$\mathbb{T}^2$ and $(\{a, b\}, k)$-$\mathbb{K}^2$ have $a = b = \frac{2k}{k-2}$.

We consider only polyhedral maps, i.e. no loops or multiple edges (1- or 2-gons), and any two faces intersect in edge, point or $\emptyset$ only.

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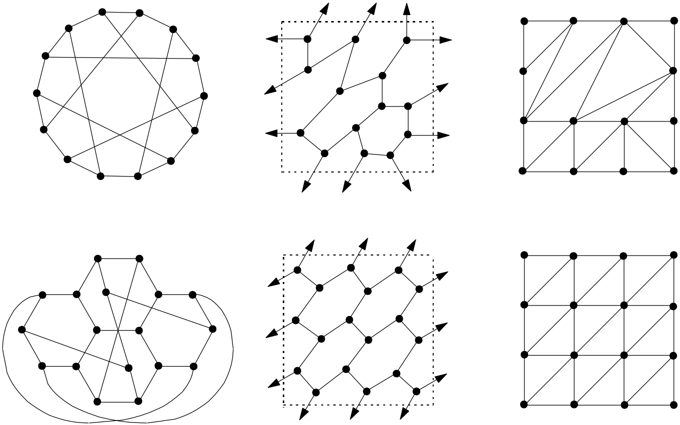
We consider only polyhedral maps, i.e. no loops or multiple edges (1- or 2-gons), and any two faces intersect in edge, point or $\emptyset$ only.

The smallest ones for (r, k)=$(6, 3), (3, 6), (4, 4)$ are embeddings

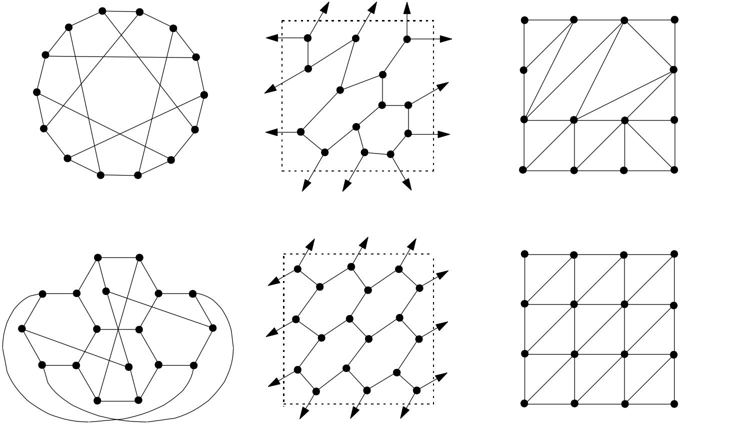
as 6-regular triangulations: K_7 and $K_{3,3,3}$ ($p_3 = 14, 18$);
as 3-regular polyhexes: Heawood graph (dual K_7) and dual $K_{3,3,3}$;
as 4-regular quadrangulations: K_5 and $K_{2,2,2}$ ($p_4 = 5, 6$).

K_5 and $K_{2,2,2}$ are also smallest $(\{3, 4\}, 4)$-$\mathbb{P}^2$ and $(\{3, 4\}, 4)$-$\mathbb{S}^2$, while K_4 is the smallest $(\{4, 6\}, 3)$-$\mathbb{P}^2$ and $(\{3, 6\}, 3)$-$\mathbb{S}^2$.

Smallest 3-regular maps on $\mathbb{T}^2$ and $\mathbb{K}^2$: duals K_7, $K_{3,3,3}$



Smallest 3-regular maps on $\mathbb{T}^2$ and $\mathbb{K}^2$: duals K_7, $K_{3,3,3}$



3-regular polyhexes on $\mathbb{T}^2$, cylinder, Möbius surface, $\mathbb{K}^2$ are $\{6^3\}$'s quotients by fixed-point-free group of isometries, generated by: two translations, a transl., a glide reflection, transl. *and* glide reflection.

8 families: symmetry groups with inversion

The point symmetry groups with inversion operation are: T_h, O_h, I_h, C_{mh}, D_{mh} with even m and D_{md}, S_{2m} with odd m. So, they are

- 9 for $\{5,6\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, S_6, T_h, D_{5d}, I_h
- 7 for $\{2,3\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, S_6, T_h
- 6 for $\{4,6\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, O_h
- 6 for $\{3,4\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{4h}, O_h
- 2 for $\{2,4\}_v$: D_{2h}, D_{4h}
- 1 for $\{3,6\}_v$: D_{2h}
- 0 for $\{2,6\}_v$ and $\{1,3\}_v$

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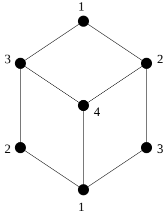
- 6 for $\{3,4\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{4h}, O_h
- 2 for $\{2,4\}_v$: D_{2h}, D_{4h}
- 1 for $\{3,6\}_v$: D_{2h}
- 0 for $\{2,6\}_v$ and $\{1,3\}_v$

(R,k)-maps on the projective plane are the antipodal quotients of centrosymmetric (R,k)-spheres; so, halving their p-vector and v.

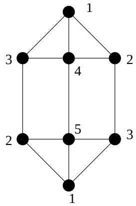
There are 6 infinite families of projective-planar $(\{a,b\},k)$-maps.

Smallest $(\{a,b\},k)$-maps on the projective plane

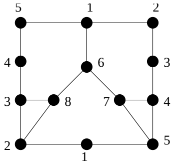
- The smallest ones for $(a,b)=(4,6),(3,4),(3,6),(5,6)$ are: K_4 (smallest $\mathbb{P}^2$-quadrangulation), K_5, 2-truncated K_4, dual K_6 (Petersen graph), i.e., the antipodal quotients of Cube $\{4,6\}_8$, $\{3,4\}_{10}(D_{4h})$, $\{3,6\}_{16}(D_{2h})$, Dodecahedron $\{5,6\}_{20}$.
- The smallest ones for $(a,b)=(2,4),(2,3)$ are points with 2, 3 loops; smallest without loops are $4\times K_2$, $6\times K_2$ but on $\mathbb{P}^2$.



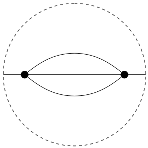
$\{4,6\}_4$



$\{3,4\}_5$



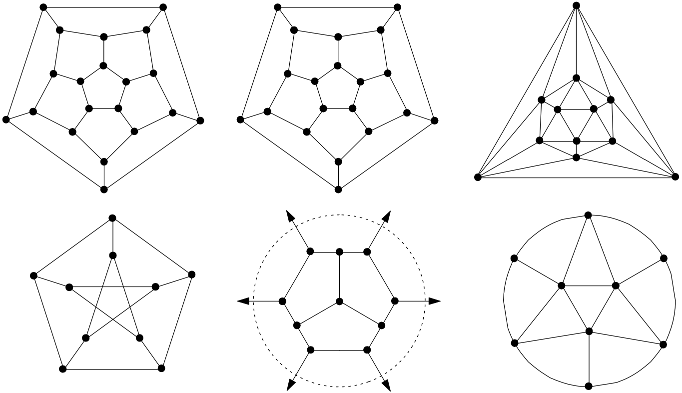
$\{3,6\}_8$



$\{2,4\}_2$

Smallest $(\{5,6\},3)$-$\mathbb{P}^2$

The Petersen graph (in positive role) is the smallest $\mathbb{P}^2$-fullerene.
Its $\mathbb{P}^2$-dual, K_6, is the antipodal quotient of Icosahedron.
K_6 is also the smallest (with 10 triangles) triangulation of $\mathbb{P}^2$.



6 families on projective plane: parameterizing

- $\{5,6\}_v$: C_i, C_{2h}, D_{2h}, S_6, D_{3d}, D_{6h}, T_h, D_{5d}, I_h
- $\{2,3\}_v$: C_i, C_{2h}, D_{2h}, S_6, D_{3d}, D_{6h}, T_h
- $\{4,6\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, O_h
- $\{3,4\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{4h}, O_h
- $\{2,4\}_v$: D_{2h}, D_{4h}
- $\{3,6\}_v$: D_{2h}

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- $\{4,6\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, O_h
- $\{3,4\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{4h}, O_h
- $\{2,4\}_v$: D_{2h}, D_{4h}
- $\{3,6\}_v$: D_{2h}

$(\{2,3\},6)$-spheres T_h and D_{6h} are $GC_{k,k}(2 \times Tetrahedron)$ and, for $k \equiv 1,2 \pmod 3$, $GC_{k,0}(6 \times K_2)$, respectively. Other spheres of blue symmetry are $GC_{k,l}$ with $l = 0, k$ from the first such sphere.

So, each of 7 blue-symmetric families is described by one natural parameter k and contains $O(\sqrt{v})$ spheres with at most vertices.

$(\{a,b\},k)$-maps on Euclidean plane and 3-space

- An $(\{a,b\},k)$-$\mathbb{E}^2$ is a k-regular tiling of $\mathbb{E}^2$ by a- and b-gons.
- $(\{a,b\},k)$-$\mathbb{E}^2$ have $p_a \leq \frac{b}{b-a}$ and $p_b = \infty$. It follows from Alexandrov, 1958: any metric on $\mathbb{E}^2$ of non-negative curvature can be realized as a metric of convex surface on $\mathbb{E}^3$.
- Consider plane metric such that all faces became regular in it. Its curvature is 0 on all interior points (faces, edges) and ≥ 0 on vertices. A convex surface is at most half-$\mathbb{S}^2$.
- There are ∞ of $(\{a,b\},k)$-$\mathbb{E}^2$'s if $2 \leq p_a \leq b$ and 1 if $p_a = 0, 1$.

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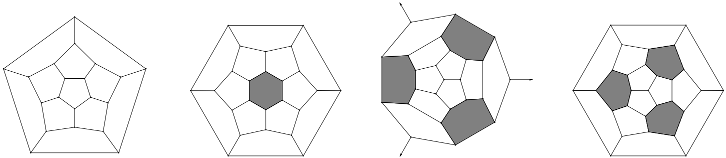
- Consider plane metric such that all faces became regular in it. Its curvature is 0 on all interior points (faces, edges) and ≥ 0 on vertices. A convex surface is at most half-$\mathbb{S}^2$.
- There are ∞ of $(\{a, b\}, k)$-$\mathbb{E}^2$'s if $2 \leq p_a \leq b$ and 1 if $p_a = 0, 1$.
- An $(\{a, b\}, k)$-$\mathbb{E}^3$ is a 3-periodic k'-regular face-to-face tiling of the Euclidean 3-space $\mathbb{E}^3$ by $(\{a, b\}, k)$-spheres.
- Next, we will mention such tilings by 4 special fullerenes, which are important in Chemistry and Crystallography. Then we consider extension of $(\{a, b\}, k)$-maps on manifolds.

X. Beyond surfaces

Frank-Kasper ($\{a, b\}, k$)-spheres and tilings

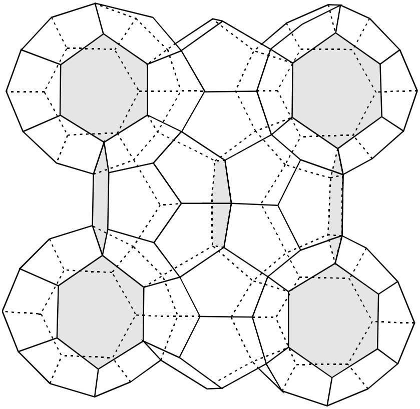
- A ($\{a, b\}, k$)-sphere is Frank-Kasper if no b-gons are adjacent.
- All cases are: smallest ones in 8 families, 3 ($\{5, 6\}, 3$)-spheres (24-, 26-, 28-vertex fullerenes), ($\{4, 6\}, 3$)-sphere $Prism_6$, 3 ($\{3, 4\}, 4$)-spheres ($APrism_4$, $APrism_3^2$, Cuboctahedron), ($\{2, 4\}, 4$)-sphere doubled square and two ($\{2, 3\}, 6$)-spheres (tripled triangle and doubled Tetrahedron).



20, I_h 24 D_{6d} 26, D_{3h} 28, T_d

FK space fullerenes

A FK space fullerene is a 3-periodic 4-regular face-to-face tiling of 3-space $\mathbb{E}^3$ by four Frank-Kasper fullerenes $\{5,6\}_v$.
They appear in crystallography of alloys, clathrate hydrates, zeolites and bubble structures. The most important, A_{15}, is below.



Other $\mathbb{E}^3$-tilings by $(\{a,b\},k)$-spheres

- An $(\{a,b\},k)$-$\mathbb{E}^3$ is a 3-periodic k'-regular face-to-face $\mathbb{E}^3$-tiling by $(\{a,b\},k)$-spheres. Some examples follow.
- Deza-Shtogrin, 1999: first known non-FK space fullerene $(\{5,6\},3)$-$\mathbb{E}^3$: 4-regular $\mathbb{E}^3$-tiling by $\{5,6\}_{20}$, $\{5,6\}_{24}$ and its elongation $\simeq \{5,6\}_{36}$ (D_{6h}) in proportion 7:2:1.

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- space cubites $(\{4,6\},3)$-$\mathbb{E}^3$: 4-, 5- and 6-regular $\mathbb{E}^3$-tilings by truncated Octahedron, by $Prism_6$ and by Cube (Voronoi of lattices $A_2 \times \mathbb{Z}$, $\mathbb{Z}^3$ and A_3^*=bcc with stars α_3, $Prism_3^*$ and β_3). Also interesting will be those with $(k'-1)$-pyramidal star.

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- space octahedrite $(\{3, 4\}, 4)$-$\mathbb{E}^3$: 8-regular (star γ_3) $\mathbb{E}^3$-tiling by Octahedron, Cuboctahedron in proportion 1:1. It is uniform Delaunay tiling of J-complex (mineral perovskite structure).
- Cf. $\mathbb{H}^3$-tilings: 6-regular $\{5, 3, 4\}$ by $\{5, 6\}_{20}$, (Löbell, 1931) by $\{5, 6\}_{24}$ and 12-reg. $\{5, 3, 5\}$ by $\{5, 6\}_{20}$, $\{4, 3, 5\}$ by Cube.

Fullerene manifolds

- Given $3 \leq a < b \leq 6$, $\{a, b\}$-manifold is a $(d-1)$-dimensional d-valent compact connected *manifold* (locally homeomorphic to $\mathbb{R}^{d-1}$) whose 2-faces are only a- or b-gonal.
- So, any i-face, $3 \leq i \leq d$, is a polytopal i-$\{a, b\}$-manifold.
- Most interesting case is $(a, b) = (5, 6)$ (fullerene manifold), when $d = 2, 3, 4, 5$ only since (Kalai, 1990) any 5-polytope has a 3- or 4-gonal 2-face.

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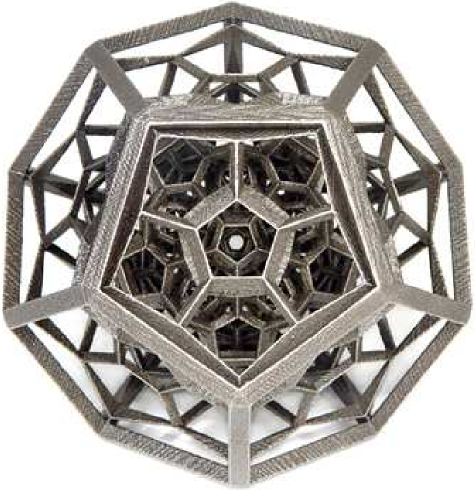
- The smallest polyhex is 6-gon on $\mathbb{T}^2$. The "greatest": $\{633\}$, the convex hull of vertices of $\{63\}$, realized on a horosphere.
- Prominent 4-fullerene (600-vertex on $\mathbb{S}^3$) is 120-cell ($\{533\}$). The "greatest" polypent: $\{5333\}$, tiling of $\mathbb{H}^4$ by 120-cells.

Projection of 120-cell in 3-space (G.Hart)



$\{533\}$: 600 vertices, 120 dodecahedral facets, $|Aut| = 14400$

4- and 5-fullerenes

- All known finite 4-fullerenes are "mutations" of 120-cell by interfering in one of ways to construct it: tubes of 120-cells, coronas, inflation-decoration method, etc.
  Some putative facets: $\simeq \{5,6\}_v(G)$ with (v, G)=$(20, I_h)$, $(24, D_{6h})$, $(26, D_3)$, $(28, T_d)$, $(30, D_{5h})$, $(32, D_{3h})$, $(36, D_{6h})$.
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- $(\{5,6\},3)$-$\mathbb{E}^3$: example of interesting infinite 4-fullerenes.
- All known 5-fullerenes come from $\{5333\}$'s by following ways.
  With 6-gons also: glue two $\{5333\}$'s on some 120-cells and delete their interiors. If it is done on only one 120-cell, it is $\mathbb{R} \times \mathbb{S}^3$ (so, simply-connected).
  Finite compact ones: the quotients of $\{5333\}$ by its symmetry group (partitioned into 120-cells) and gluings of them.

Quotient d-fullerenes

- Selberg, 1960, Borel, 1963: if a discrete group of motions of a symmetric space has a compact fundamental domain, then it has a torsion-free normal subgroup of finite index.
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- Exp. 1: Polyhexes on $\mathbb{T}^2$, cylinder, Möbius surface and $\mathbb{K}^2$ are the quotients of $\{6^3\}$ by discontinuous fixed-point-free group of isometries, generated by: 2 translations, a translation, a glide reflection, translation *and* glide reflection, respectively.

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- Exp 2: Poincaré dodecahedral space: the quotient of 120-cell by I_h ; so, its f-vector is $(5, 10, 6, 1) = \frac{1}{120}$f(120-cell).
- Cf. 6-, 12-regular $\mathbb{H}^3$-tilings $\{5, 3, 4\}$, $\{5, 3, 5\}$ by $\{5, 6\}_{20}$ and 6-regular $\mathbb{H}^3$-tiling by (right-angled) $\{5, 6\}_{24}$.
  Seifert-Weber, 1933 and Löbell, 1931 spaces are quotients of last 2 with f-vectors $(1, 6, p_5{=}6, 1)$, $(24, 72, 48{+}8{=}p_5{+}p_6, 8)$.