Weavings of the Cube and Other Polyhedra

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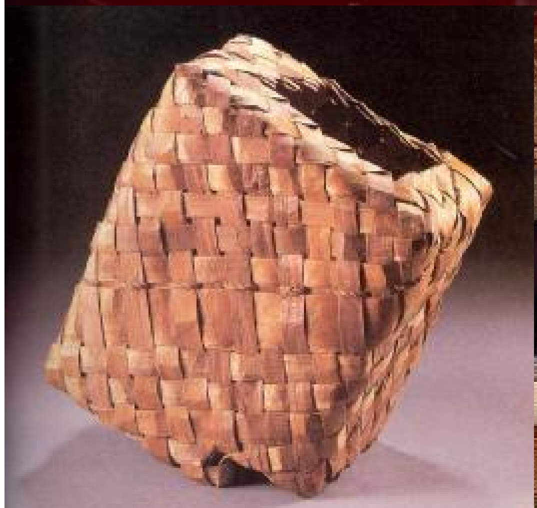
Contents

- Basket weaving in practice
- Weaving of the cube
- Geometrical approach
- Graph-theory based approach
- Extensions
- Conclusions

BASKET WEAVING IN PRACTICE

Open baskets



Japan pavilion, Aichi EXPO 2005

Japanese Government + Nihon-Sekkei Inc., Kumagaigumi Co.Ltd

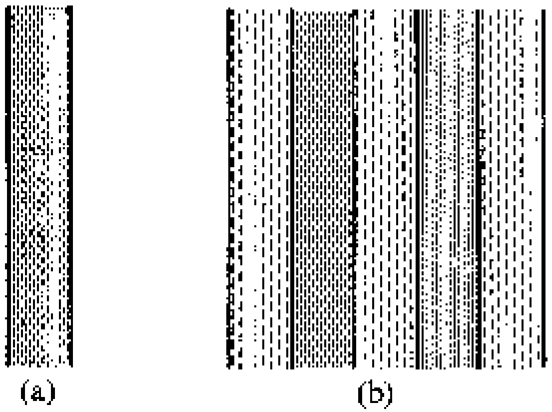
Closed baskets

Closed baskets (balls)

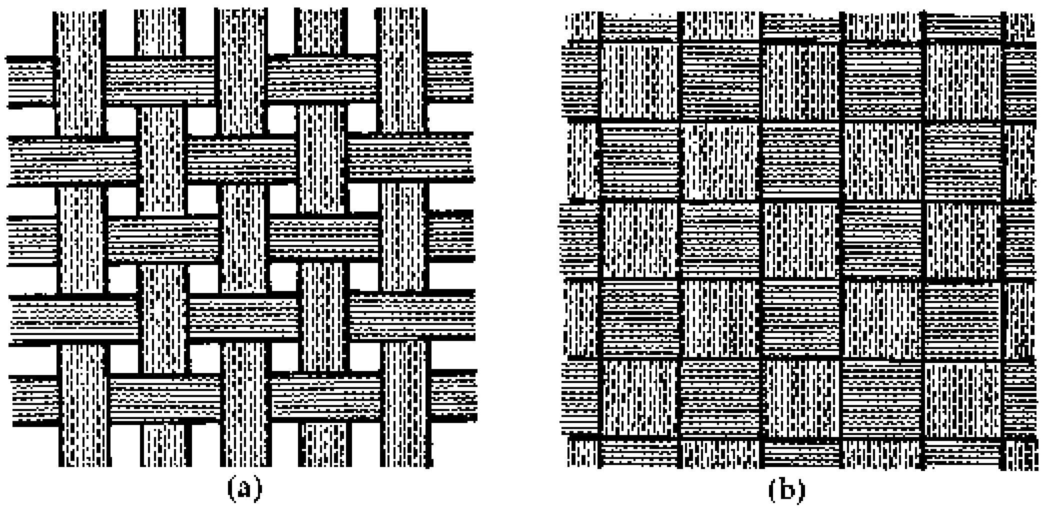
WEAVING THE CUBE

2-way 2-fold weaving

in the plane



. (a) single strand; (b) parallel strands.



on a polyhedron

Cube, parallel

Cube, 45 degrees

Felicity Wood

Skew weaving

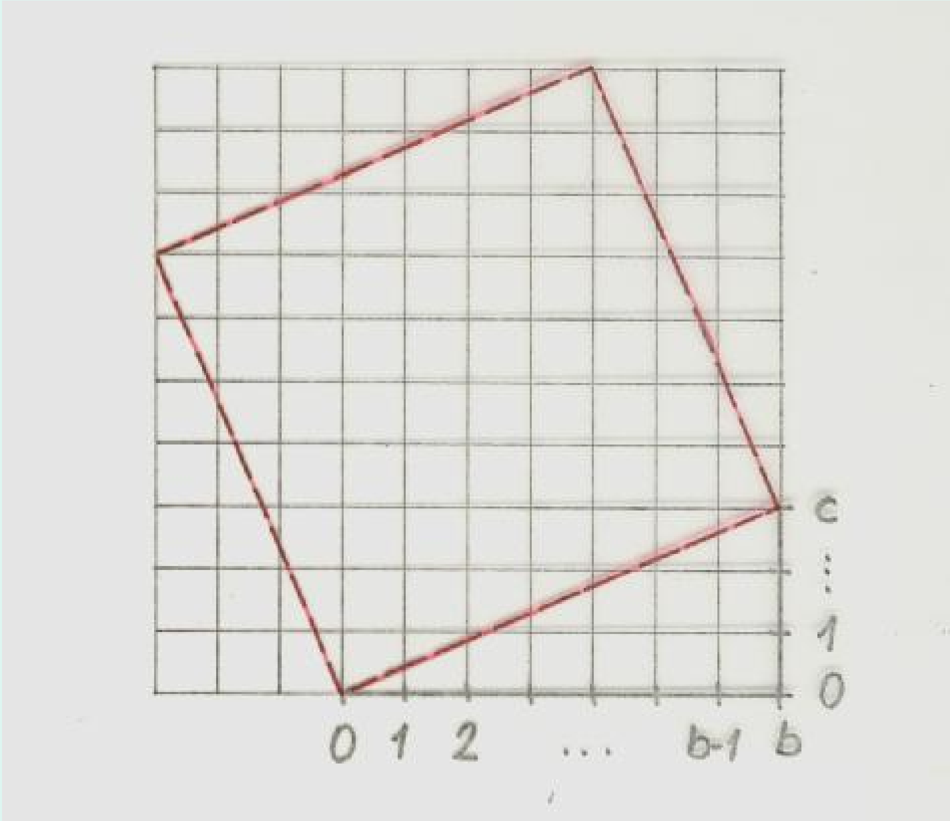
Felicity Wood, 2006

GEOMETRICAL APPROACH

Definition

We talk about *wrapping*, if the physical weave is simplified to a *double cover*, where the up-down relationship of the strands has been “flattened out”.

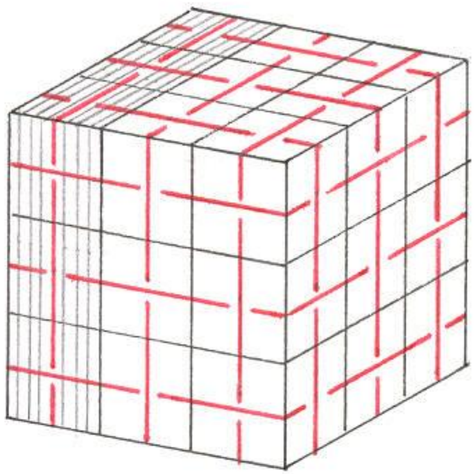
The Coxeter notation



$\{4,3+\}_{b,c}$

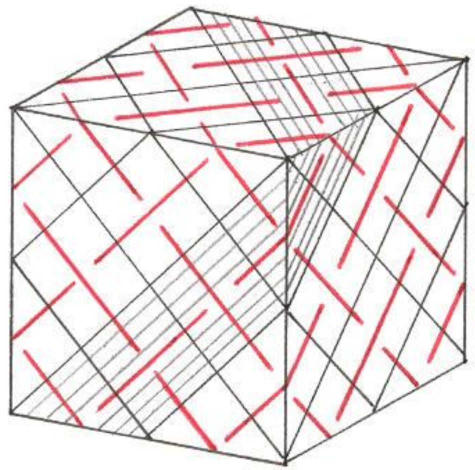
$S = b^2 + c^2$

The three classes of cube wrapping



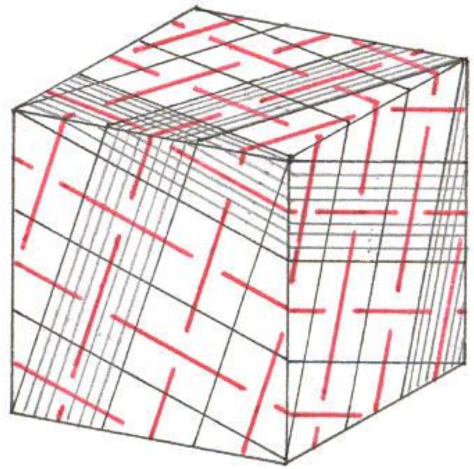
Class I

$b = 0$ or $c = 0$



Class II

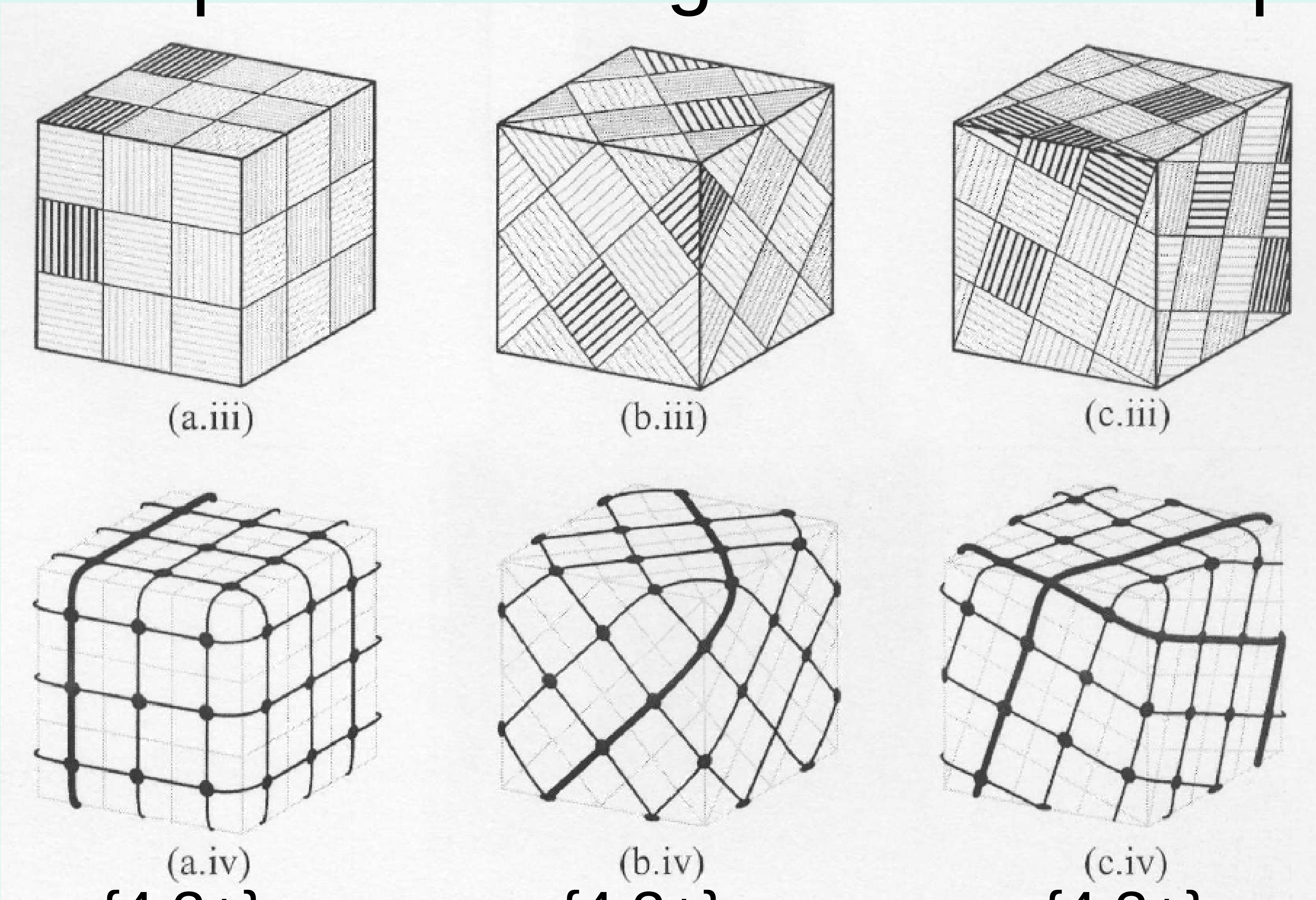
$b = c$



Class III

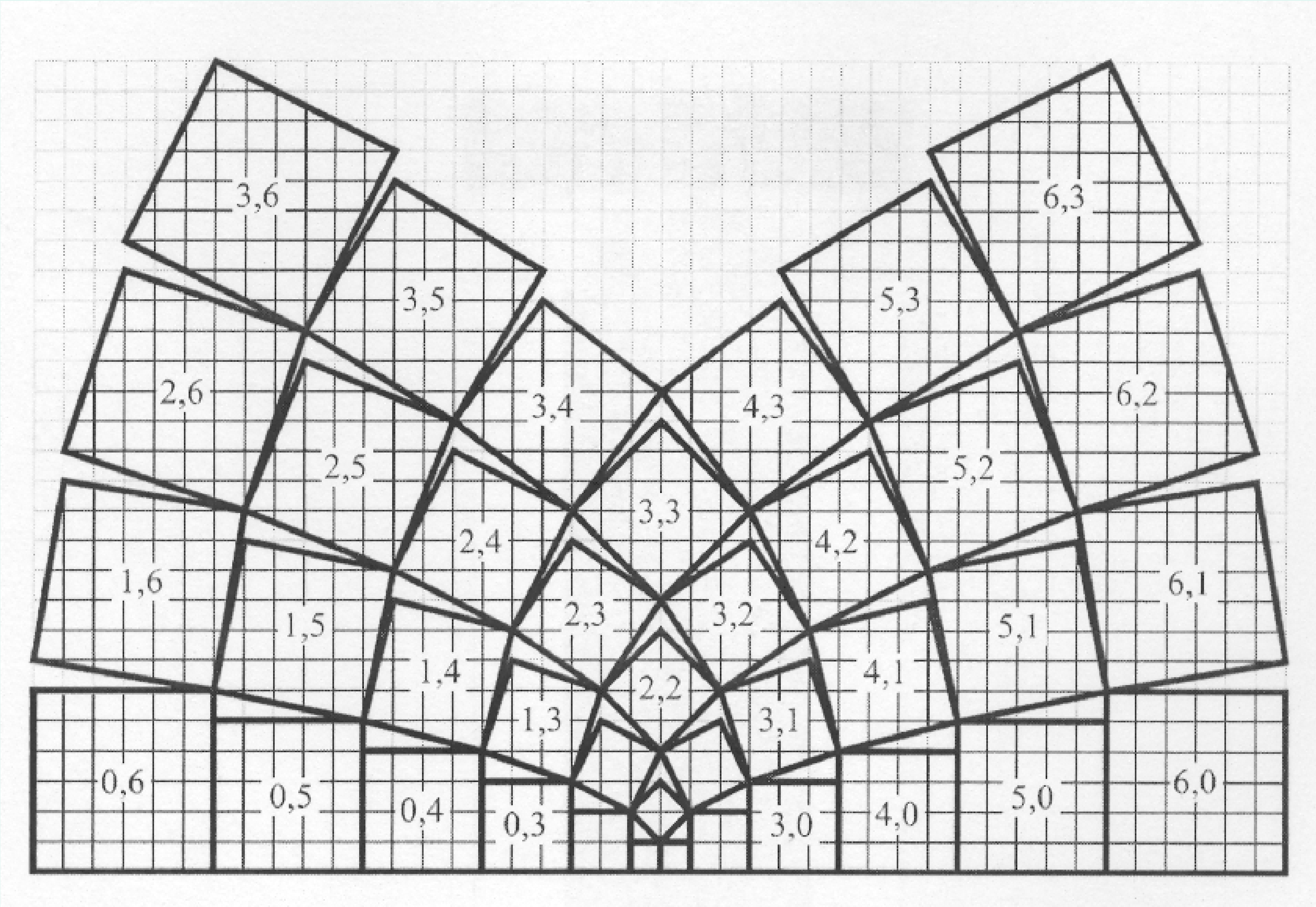
$b \neq c$, $b \neq 0$, $c \neq 0$

Complete weavings and dual maps



$\{4,3+\}_{3,0}$ $\{4,3+\}_{2,2}$ $\{4,3+\}_{3,1}$

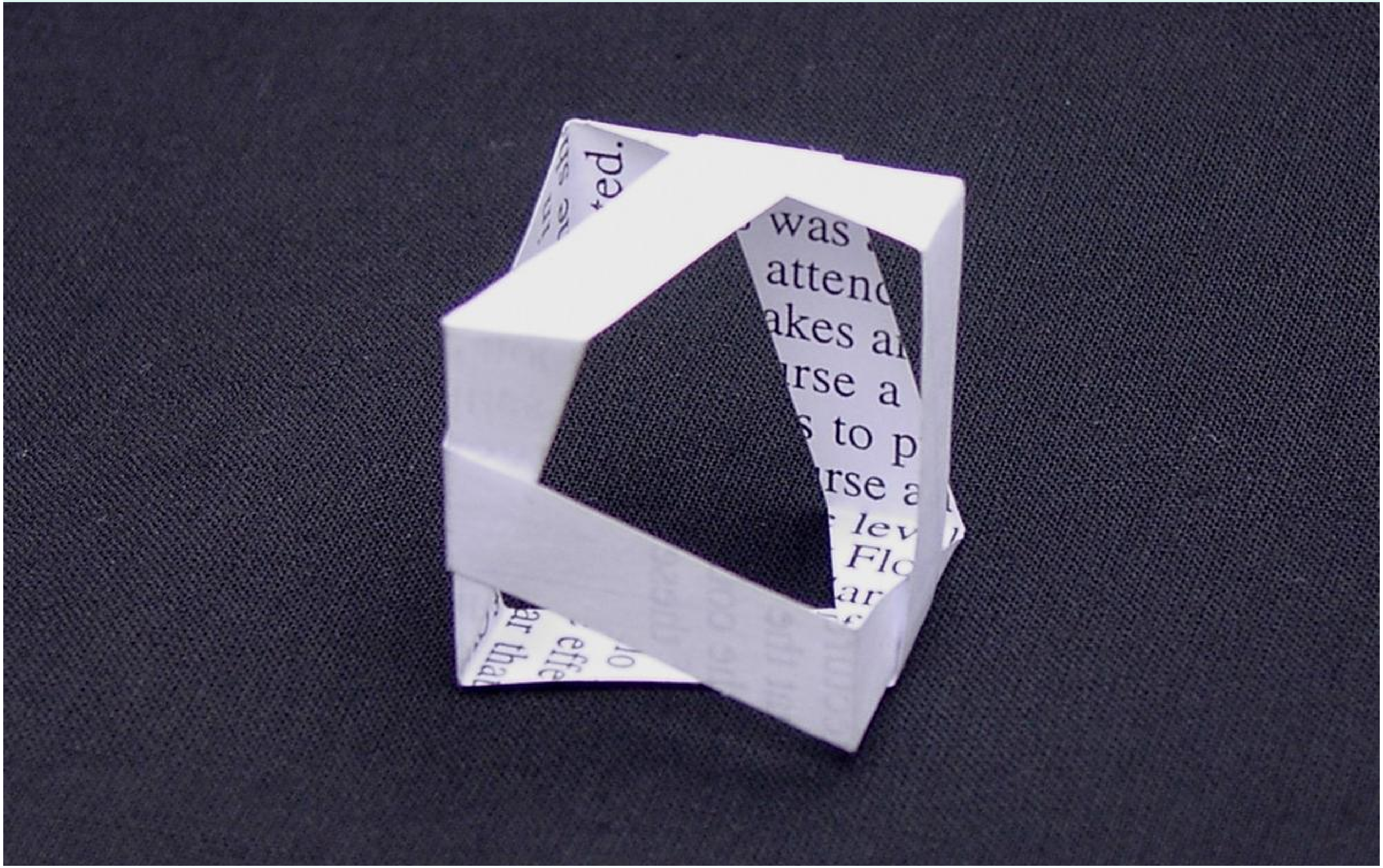
Tiling of the faces of the cube



Properties of strands

- The midline of a strand is a geodesic on the surface of the cube.
- Since *b* and *c* are positive integers, the midlines form closed geodesics (loops).
- If *b* and *c* are co-prime then all loops are congruent.
- For any given pair *b*, *c*, the lengths of all loops are equal.

One loop for $b = 3$, $c = 1$



Questions for given *b*, *c*

- How many strands are there?
- How large a torsion (twist) does a strand have? (What is the linking number of the two boundary lines of a strand?)
- What sort of knot does a strand have?

Number of loops, *n*

c \ b	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
16	48	3	6	3	12	6	6	3	48	3	6	3	12	3	12	3	64
15	45	4	6	12	3	20	9	4	3	12	15	4	18	4	3	60	3
14	42	6	8	3	12	3	8	42	6	6	8	3	6	6	56	3	12
13	39	4	3	4	3	4	3	4	6	4	3	4	3	52	6	4	3
12	36	3	12	9	16	3	36	3	12	9	6	3	48	3	6	18	12
11	33	4	3	4	6	4	3	4	6	4	6	44	3	4	3	4	3
10	30	6	8	3	6	30	8	6	12	3	40	6	6	3	8	15	6
9	27	4	6	12	3	4	9	4	3	36	3	4	9	4	6	12	3
8	24	3	6	3	24	3	6	6	32	3	12	6	12	6	6	3	48
7	21	4	6	4	3	4	3	28	6	4	6	4	3	4	42	4	3
6	18	6	8	18	6	3	24	3	6	9	8	3	36	3	8	9	6
5	15	4	3	4	6	20	3	4	3	4	30	4	3	4	3	20	6
4	12	3	12	3	16	6	6	3	24	3	6	6	16	3	12	3	12
3	9	4	3	12	3	4	18	4	3	12	3	4	9	4	3	12	3
2	6	6	8	3	12	3	8	6	6	6	8	3	12	3	8	6	6
1	3	4	6	4	3	4	6	4	3	4	6	4	3	4	6	4	3
0	0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48
c b	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Observations

- The table is symmetric with respect to the line $c = b$.
- For given c, the sequence is periodic with period $p = 4c$, and the i-th period is symmetric with respect to the point $b = 4c(i - 1/2)$.
- By periodicity, if $b \equiv b_1 \pmod{4c}$, then $n(b,c) = n(b_1,c)$.
- If b,c are co-prime, then $n = 3, 4, 6$.
- If $b = kb_1$, $c = kc_1$, b_1 and c_1 are co-prime, $k > 0$, then $n(b,c) = kn(b_1,c_1)$.

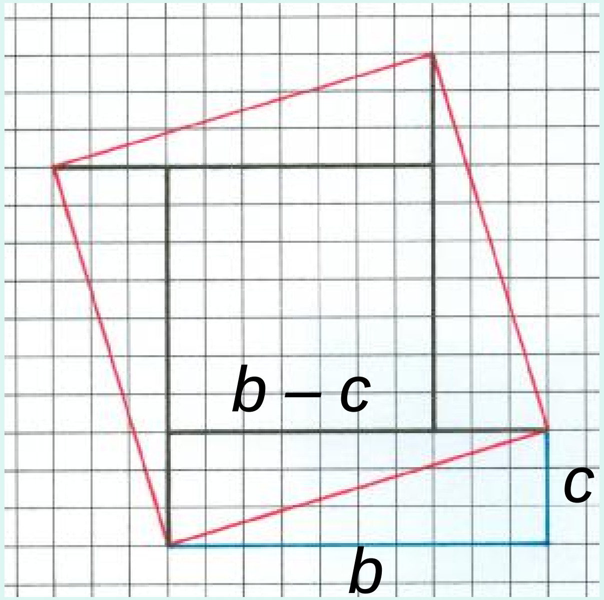
b and *c* co-prime

16		**3**		**3**		**6**		**3**		**3**		**3**		**3**		**3**	
15		**4**	**6**		**3**			**4**	**3**			**4**		**4**	**3**		**3**
14		**6**		**3**		**3**				**6**		**3**		**6**		**3**	
13		**4**	**3**	**4**	**3**	**4**	**3**	**4**	**6**	**4**	**3**	**4**	**3**		**6**	**4**	**3**
12		**3**				**3**		**3**				**3**		**3**			
11		**4**	**3**	**4**	**6**	**4**	**3**	**4**	**6**	**4**	**6**		**3**	**4**	**3**	**4**	**3**
10		**6**		**3**				**6**		**3**		**6**		**3**			
9		**4**	**6**		**3**	**4**		**4**	**3**		**3**	**4**		**4**	**6**		**3**
8		**3**		**3**		**3**		**6**		**3**		**6**		**6**		**3**	
7		**4**	**6**	**4**	**3**	**4**	**3**		**6**	**4**	**6**	**4**	**3**	**4**		**4**	**3**
6		**6**				**3**		**3**				**3**		**3**			
5		**4**	**3**	**4**	**6**		**3**	**4**	**3**	**4**		**4**	**3**	**4**	**3**		**6**
4		**3**		**3**		**6**		**3**		**3**		**6**		**3**		**3**	
3		**4**	**3**		**3**	**4**		**4**	**3**		**3**	**4**		**4**	**3**		**3**
2		**6**		**3**		**3**		**6**		**6**		**3**		**3**		**6**	
1	**3**	**4**	**6**	**4**	**3**	**4**	**6**	**4**	**3**	**4**	**6**	**4**	**3**	**4**	**6**	**4**	**3**
0		**3**															
c b	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**

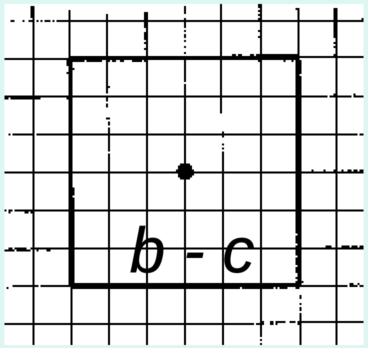
b or *c* even

c \ *b*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
16		3		3		6		3		3		3		3		3	
15		4	6		3			4	3			4		4	3		3
14		6		3		3				6		3		6		3	
13		4	3	4	3	4	3	4	6	4	3	4	3		6	4	3
12		3				3		3				3		3			
11		4	3	4	6	4	3	4	6	4	6		3	4	3	4	3
10		6		3				6		3		6		3			
9		4	6		3	4		4	3		3	4		4	6		3
8		3		3		3		6		3		6		6		3	
7		4	6	4	3	4	3		6	4	6	4	3	4		4	3
6		6				3		3				3		3			
5		4	3	4	6		3	4	3	4		4	3	4	3		6
4		3		3		6		3		3		6		3		3	
3		4	3		3	4		4	3		3	4		4	3		3
2		6		3		3		6		6		3		3		6	
1	3	4	6	4	3	4	6	4	3	4	6	4	3	4	6	4	3
0		3															

Why is n = 3, 4, 6 ?

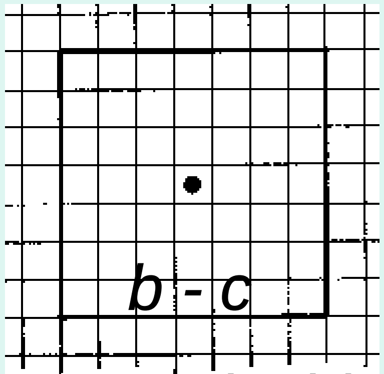


(1) $b - c$ is even



Face centre coincides with a a vertex of the tessellation

(2) $b - c$ is odd



Face centre coincides with the centre of a small square

GRAPH-THEORY BASED APPROACH

Basic terms and statements

(Deza and Shtogrin, 2003)

- 4-valent polyhedra having 8 triangular faces, while all other faces are quadrangular, are called *octahedrites*
- a *central circuit* of an octahedrite enters and leaves any given vertex by opposite edges
- octahedrites of *octahedral symmetry* are duals of tilings on the cube

- any octahedrite is a projection of an *alternating link* whose components correspond to central circuits
- a *rail-road* is a circuit of quadrangular faces, in which every quadrangle is adjacent to two of its neighbours on opposite edges
- an octahedrite with no rail-road is *irreducible*

Octahedrites Wrapping

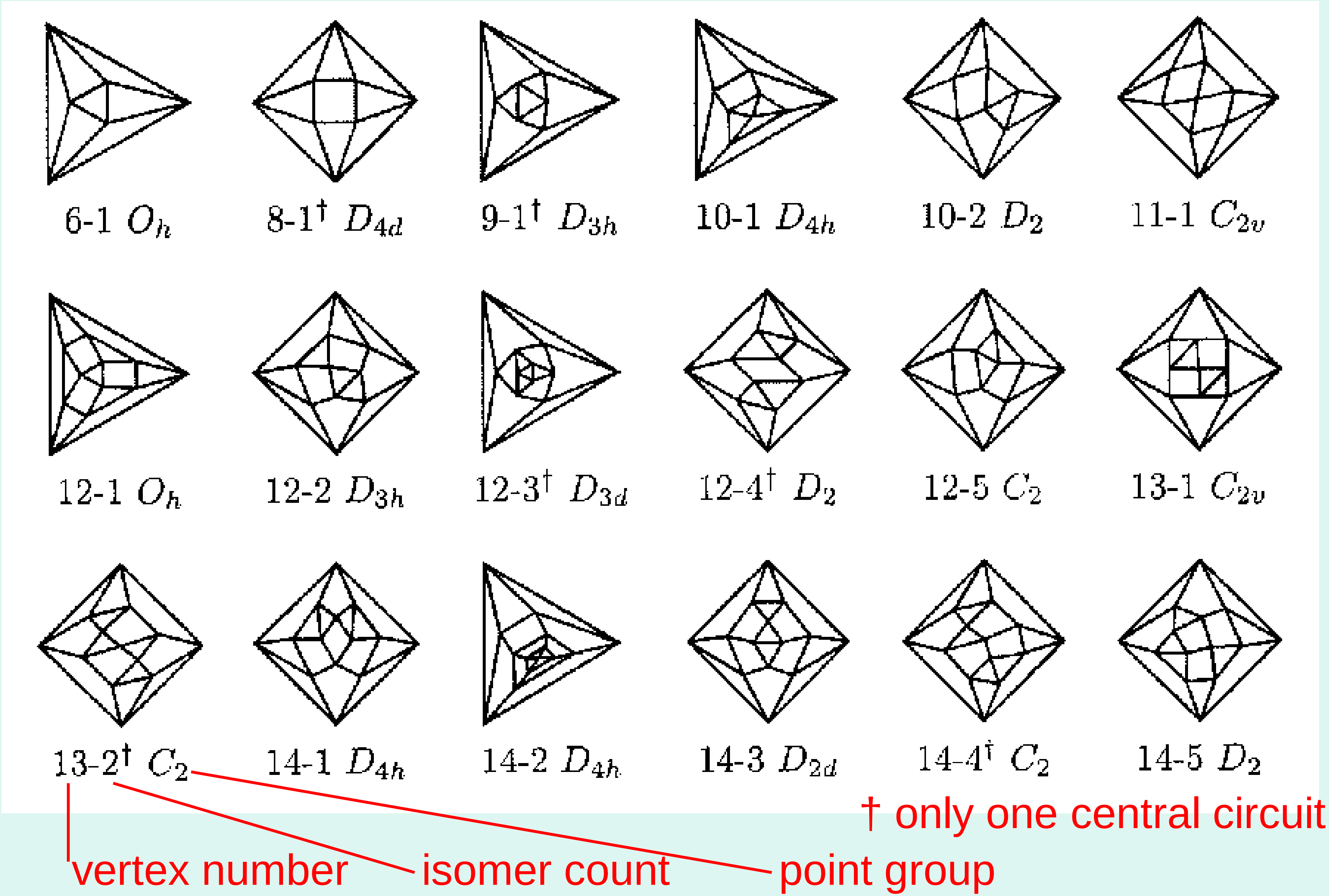
- Dual of octahedrite → square tiling
- Alternating link → weaving
- Central circuit → midline of a strand
- Rail-road → adjacent parallel strands
- Irreducible octahedrite → *b*,*c* are co-prime

Octahedrite 12–1 O_h (red) and dual

Realization by Felicity Wood

The smallest octahedrites ...

after Deza and Shtogrin (2003)



Wrappings based on octahedrites

6–1 O_h

8–1 D_{4d}

9–1 D_{3h}

10–1 D_{4h}

10–1 D_2

11–1 C_{2v}

Wrappings of the cube

6–1 O_h	12–1 O_h	24 O_h	30 O
$\{b,c\}=\{1,0\}$	$\{1,1\}$	$\{2,0\}$	$\{2,1\}$

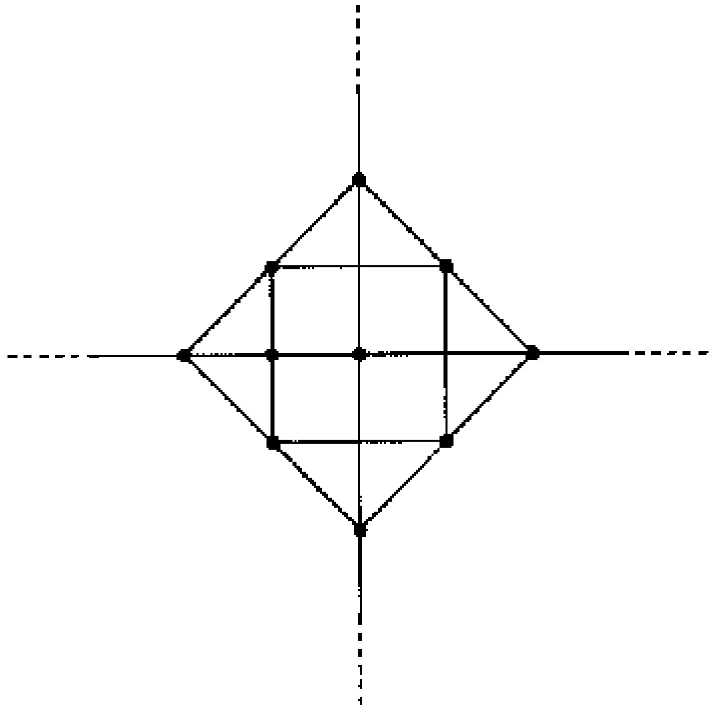
Wrappings of the square antiprism

where triangular faces are right isosceles triangles

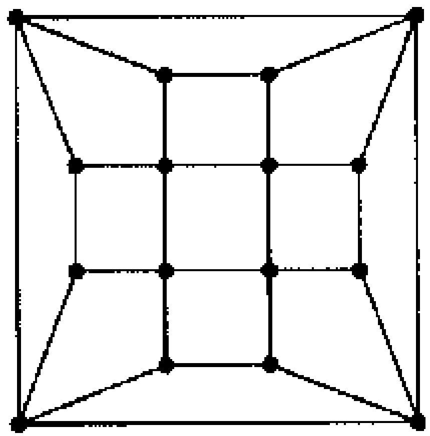
{*b*,*c*}={1,0} {1,1} {2,0} {2,1}

The numbers *b*, *c* are related to the short sides of the triangles

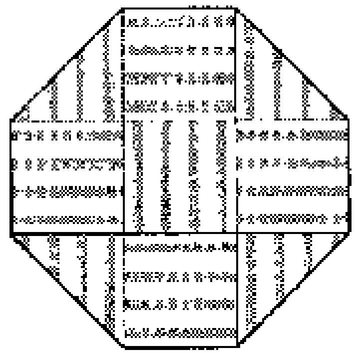
Wrapping of an octagon



Octahedrite 14–1 D_{4h}



Its dual



Wrapping of
a two-layer
octagon

EXTENSIONS

Symmetrically crinkled structures



Wrapping based on dualising 4-valent polyhedra with quadrangular, pentagonal and hexagonal faces (square, pentagonal, hexagonal antiprisms)

i-hedrites, definition

(Deza et *al.* 2003)

4-valent planar graphs with digonal, triangular and quadrangular faces, obeying the constraints

$$f_2 + f_3 = i, \quad f_2 = 8 - i, \quad i = 4, \ldots, 8$$

are called *i-hedrites.*

Different realizations of wrapping based on dualising an *i*-hedrite



$f_3 = 0, \qquad f_2 = i = 4$

Convex realization of wrapping based on dualising an *i*-hedrite

$f_3 = 0, \quad f_2 = i = 4$

Ongoing

- What polyhedra can be wrapped?
- What convex realisations can be achieved?
- Alexandrov Theorem
- From dual octahedrites, can we achieve wrappings of *all* the 257 8-vertex polyhedra + octagon?

… watch this space! …

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